8.7 Determinants and Cramer's Rule

<u>Defn</u>: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix

then the <u>determinant</u> of matrix A,

denoted $\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

<u>Cramer's</u> Rule for 2 equations and 2 variables

<u>Methods</u>
1. Graphing
2. Substitution
3. Elimination
4. Elimination with matrices
5. Cramer's Rule using determinant

<u>ex</u>: $\begin{cases} 2x + 4y = 1 \\ 3x + 5y = 2 \end{cases}$

Calculate three determinants

$D = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix}$ = determinant of the coefficient matrix

$= (2)(5) - (3)(4) = 10 - 12$

$= -2$

D_x = determinant where we <u>replace</u> the x-coefficients with the constants.

$$= \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = (1)(5) - (2)(4)$$

$$= 5 - 8 = -3$$

D_y = determinant gotten by replacing the y-coefficients with the constants

$$= \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = (2)(2) - (3)(1)$$

$$= 4 - 3 = 1$$

The solution is (x, y) where

$$x = \frac{D_x}{D} = \frac{-3}{-2} = \frac{3}{2}$$

$$y = \frac{D_y}{D} = \frac{1}{-2} = -\frac{1}{2}$$

so $(x, y) = \left(\frac{3}{2}, -\frac{1}{2}\right)$ ✓

<u>Remark</u>: There is a version of Cramer's Rule for 3 eqns and 3 variables, it's so tedious I don't recommend it.

Use Cramer's Rule to solve

ex: $2x + 3y = 5$
$6x + 9y = 1$

$$D = \begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = (2)(9) - (6)(3) = 0$$

Whoa!!! This system will be inconsistent (no solution) or dependent (infinitely many solutions)

which?

i) If BOTH D_x and D_y are zero, ~~the~~ the system is dependent.

ii) Otherwise (if one of D_x or D_y are not zero) the system is inconsistent.

$$D_x = \begin{vmatrix} 5 & 3 \\ 1 & 9 \end{vmatrix} = (5)(9) - (1)(3) = 42 \neq 0$$

$$D_y = \begin{vmatrix} 2 & 5 \\ 6 & 1 \end{vmatrix} = (2)(1) - (6)(5) = 2 - 30 = -28 \neq 0$$

$\therefore$ The system is $\boxed{\text{inconsistent}}$. no solution.

Use Cramer's Rule

ex: $3x + y = 0$
$-2x + 5y = 0$

$$D = \begin{vmatrix} 3 & 1 \\ -2 & 5 \end{vmatrix} = (3)(5) - (-2)(1)$$
$$= 15 - (-2) = 17 \neq 0$$

The will be one solution.

$$D_x = \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} = (0)(5) - (0)(1) = 0$$

$$D_y = \begin{vmatrix} 3 & 0 \\ -2 & 0 \end{vmatrix} = (3)(0) - (-2)(0) = 0$$

$$x = \frac{D_x}{D} = \frac{0}{17} = 0 \qquad y = \frac{D_y}{D} = \frac{0}{17} = 0$$

Soln: $(x, y) = (0, 0)$

Calculate the following determinant:

ex: [3×3 determinant – by a method NOT in the textbook]

$$
\begin{array}{ccccc}
 & & 20 & 14 & 18 \\
2 & 1 & 4 & 2 & 1 \\
3 & 5 & 1 & 3 & 5 \\
1 & 7 & 6 & 1 & 7 \\
 & & 60 & 1 & 84
\end{array}
$$

$$= 60 + 1 + 84 - 20 - 14 - 18$$

$$= 145 - 52 = \boxed{93}$$

Application: Consider the system of equations:

$$
\begin{cases}
2x + y + 4z = 1 \\
3x + 5y + z = -7 \\
x + 7y + 6z = -3
\end{cases}
$$

~~Is this system consistent?~~ Is there exactly one solution?

Yes, because $D = 93 \neq 0$.

Remark: You should try to understand the 3×3 version of Cramer's Rule by reading on your own, but you will not be tested on it.

9.1 and 9.2 Inequalities and compound inequalities in one variable.

ex: $7 < 15$ so $3(7) < 3(15)$

$21 < 45$

$-2 < 5$ so $10(-2) < 10(5)$

$-20 < 50$

$3 < 7$ but $(-2)(3) > (-2)(7)$

$-6 > -14$

$-2 < 5$ but $(-10)(-2) > (-10)(5)$

$20 > -15$

$-4 < -1$ but $(-3)(-4) > (-3)(-1)$

$12 > 3$

ex: Solve $3x + 1 > 13$

$\quad -1 \quad\quad -1$

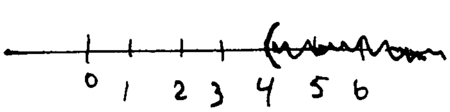
$3x > 12$

$\frac{3x}{3} > \frac{12}{3}$

$x > 4$

Three ways to express the answer:

(1) Graph



(2) Interval notation

$(4, \infty)$

(3) Set-builder notation

$\{x \mid x > 4\}$

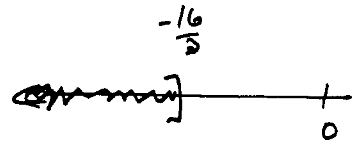
ex: $-3x + 5 \geq 21$

$\quad\quad -5 \quad -5$

$-3x \geq 16$

$\dfrac{-3x}{-3} \leq \dfrac{16}{-3}$

$x \leq -\dfrac{16}{3}$



$\left(-\infty, -\dfrac{16}{3}\right]$

$\left\{x \mid x \leq -\dfrac{16}{3}\right\}$

Solve. Express the answer using interval notation:

24) $8x - 3(3x+2) - 5 \geq 3(x+4) - 2x$

$8x - 9x - 6 - 5 \geq 3x + 12 - 2x$

$-x - 11 \geq x + 12$

$\quad +11 \quad\quad +11$

$-x \geq x + 23$

$-x \quad -x$

$-2x \geq 23$

$\dfrac{-2x}{-2} \leq \dfrac{23}{-2}$

$x \leq -\dfrac{23}{2} = -11.5$

9.2:
"Union" and "Intersection" of Sets

ex: a) Find the intersection of these two sets

$A = \{0\ 2\ 4\ 6\ 8\ 10\ 12\}$ $\quad B = \{0\ 3\ 6\ 9\ 12\}$

intersection $= A \cap B = \{0\ 6\ 12\}$

NOTE: A number appears in the intersection if it appears in set A and it appears in set B.

b) Find the union of sets A and B

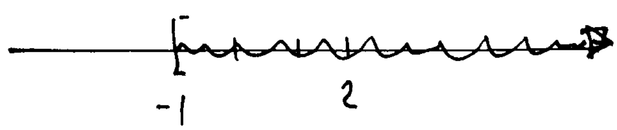
union $= A \cup B = \{0\ 2\ 3\ 4\ 6\ 8\ 9\ 10\ 12\}$

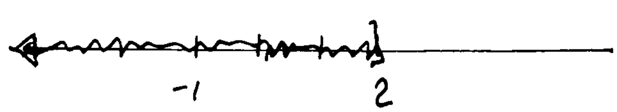
NOTE: A number appears in the union if it appears in set A OR it appears in set B.

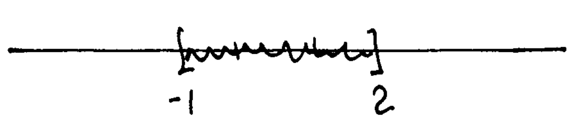
Determine what set this is by graphing.

ex: $\underbrace{\{x \mid x \geq -1\}}_{A} \cap \underbrace{\{x \mid x \leq 2\}}_{B}$

a)

A  $[-1, \infty)$

B  $(-\infty, 2]$

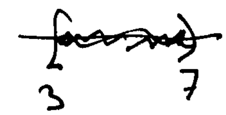
$A \cap B$  $[-1, 2]$

b) $A \cup B$  $(-\infty, \infty)$

ex: (new notation) $3 \le x < 7$

is defined to mean $3 \le x$ AND $x < 7$

Write the solution set in interval notation $[3, 7)$



ex: Can you find a number which satisfies

$$5 < x < 1 \quad ?$$

No because this means $5 < x$ AND $x < 1$

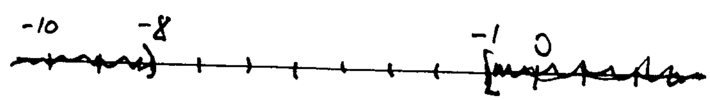
Solution set = empty set = $\emptyset$

ex: Solve and graph each solution set

58) $x + 5 < -3$ OR $x + 5 \ge 4$

$\quad -5 \quad -5 \qquad\qquad -5 \quad -5$

$x < -8$ OR $x \ge -1$



solution set $= (-\infty, -8) \cup [-1, \infty)$