10.8 Complex Numbers

Define $i = \sqrt{-1}$ so that

$$\boxed{i^2 = -1}$$

Remark: (1) Note that i is a constant just like π.

(2) Note that i is not a real number.

ex: $\sqrt{-49} = \sqrt{49}\sqrt{-1} = 7i$

check: $(7i)^2 = 7^2 \cdot i^2 = 49(-1) = -49$.

Defn. A complex number is a number that has the form

$$\boxed{a+bi}$$

where a and b are real numbers.

examples: $3+4i$, $-7+2i$, $\frac{1}{2}+\frac{\sqrt{3}}{2}i$,

$16i = 0+16i$; $3 = 3+0i$

Remark: Every real number is a complex number.

Arithmetic of complex numbers:

ex [add]

$$(3+5i)+(14-7i)$$
$$= 3+14+5i+(-7i)$$
$$= 17-2i$$

ex: $\left(\frac{1}{2}+\frac{3}{2}i\right)+\left(\frac{7}{2}+i\right) = 2+\frac{5}{2}i$

ex [subtract]: $(6+5i)-(2+3i)$
$$= 6+5i-2-3i$$
$$= 4+2i$$

ex: [multiply] $(3i)(5i) = 3\cdot 5\cdot i\cdot i = 15i^2 = 15\cdot(-1)$
$$= -15$$

ex $(10)(7i) = 70i$

ex: $10(3+4i) = 30+40i$

ex: $2i(3+4i) = 6i+8i^2$
$$= 6i+8(-1) = -8+6i$$

ex: $(5+2i)(3+4i) = (5)(3)+(5)(4i)$
$$+(3)(2i)+(2i)(4i)$$
$$= 15+20i+6i+8i^2$$
$$= 15+26i-8 = 7+26i$$

ex [powers of i]

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$i^6 = i^4 \cdot i^2 = -1$

$i^7 = i^4 \cdot i^3 = -i$

$i^8 = i^4 \cdot i^4 = 1$

$i^9 = i$

$i^{10} = -1$

$i^{11} = -i$

$i^{12} = 1$

$\therefore \quad i^{2015} = -i$

Reason: $i^{2015} = i^{2012+3} = i^{2012} \cdot i^3$

$= (i^4)^{503} \cdot i^3$

$= 1^{503} \cdot (-i) = -i$

ex: [reciprocals]

$$\frac{1}{3+4i} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-4i}{3^2-(4i)^2}$$

$$= \frac{3-4i}{9-16i^2} = \frac{3-4i}{9-16(-1)}$$

$$= \frac{3+4i}{9+16} = \frac{3+4i}{25}$$

$$= \frac{3}{25} + \frac{4}{25}i$$