10.8 Complex numbers (cont'd)

Warmup

(1) $(3+4i)+(2+5i) = 5+9i$

(2) $(3+4i)-(2+5i) = 3+4i-2-5i = 1-i$

(3) $(3+4i)\cdot(2+5i) = 6+15i+8i+20i^2$
$= 6+23i+20(-1)$
$= -14+23i$

(4) $1+2i+3i^2+4i^3+5i^4+6i^5$
$= 1+2i-3-4i+5+6i$
$= 3+4i$

Useful formulas

(1) $(a+b)^2 = a^2+2ab+b^2$
(2) $(a-b)^2 = a^2-2ab+b^2$
(3) $(a-b)(a+b) = a^2-b^2$

ex: $(3+4i)^2 = 3^2+2\cdot3\cdot4i+(4i)^2$
$= 9+24i-16$
$= -7+24i$

<u>ex</u>: [product of complex number with its Conjugate]

$$(3-4i)(3+4i) = 3^2-(4i)^2$$
$$= 9-(-16) = 9+16 = 25$$

Try this!
<u>ex</u>: $(2+5i)\cdot(2-5i) = 2^2-(5i)^2$
$$= 4-(-25) = 4+25 = 29$$

<u>more generally</u>: If a and b are real numbers,

$$(a+bi)\cdot(a-bi) = a^2-(bi)^2$$
$$= a^2 - b^2\cdot i^2 = a^2 - b^2\cdot(-1)$$
$$= a^2-(-b^2) = a^2+b^2$$

Aha! A <u>positive</u> real number (unless $a=b=0$, then zero)

<u>ex</u> [Reciprocals and correction of Tuesday's notes]

$$\frac{1}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{3-4i}{3^2+4^2} = \frac{3-4i}{25}$$
$$= \frac{3}{25}-\frac{4}{25}i$$

Ex [Division] $\dfrac{-5+10i}{3+4i} \cdot \dfrac{3-4i}{3-4i}$

$$= \frac{-15+20i+30i-40i^2}{9-16i^2} = \frac{-15+50i+40}{9+16}$$

$$= \frac{25+50i}{25} = \frac{25}{25} + \frac{50}{25}i = 1+2i$$

Try this ex: $\dfrac{8-4i}{1-3i} \cdot \dfrac{1+3i}{1+3i} = \dfrac{8+24i-4i-12i^2}{1-9i^2}$

$$= \frac{8+20i+12}{1+9} = \frac{20+20i}{10} = \frac{20}{10} + \frac{20}{10}i$$
$$= 2+2i$$

ex: Show that the quadratic equation

$$x^2 - 2x + 5 = 0$$

is satisfied by $x = 1+2i$. (Or show it's not satisfied)

Scratch work: $x^2 = (1+2i)^2 = 1^2 + 2\cdot 1\cdot 2i + (2i)^2 = 1+4i+4i^2$
$= 1+4i-4 = -3+4i$

$-2x = -2(1+2i) = -2-4i$

$5 = 5$

$x^2-2x+5 = -3+4i-2-4i+5 = (-3-2+5)+(4i-4i)$
$= 0$ ✓

11.1 Quadratic Equations

Remark Naming conventions for polynomials

Degree	Name	Example
0	Constant polynomial	5
1	Linear "	$2x-1$
2	Quadratic "	$x^2-8x+15$
3	Cubic "	x^3+3x^2+3x+1
4	Quartic "	$10x^4-8x^2+17$
⋮		

Defn: A quadratic equation is one in which one (or both) sides of the equation are quadratic polynomials:

examples: $x^2-8x+15=0$, $2x-3=x^2+15$

$(x+2)(x-3)=0$, $(x+1)(x+10)=x-5$.

ex: $x^2-8x+15=0$ ← "method of solving a quadratic equation by factoring."

$(x-5)(x-3)=0$

$x-5=0$ OR $x-3=0$

$x=5$ OR $x=3$

Remark: We will learn three methods for solving quadratic equations:

1. Factoring.
2. Completing the square.
3. Quadratic formula.

Toward "completing the square" method.

ex: $x^2 = 49$

$x^2 - 49 = 0$

$(x-7)(x+7) = 0$

$x - 7 = 0$ or $x + 7 = 0$

$x = 7$ or $x = -7$

Shortcut: $x^2 = 49 \Rightarrow x = \pm\sqrt{49} = \pm 7$

meaning: $x = 7$ or $x = -7$

ex: $x^2 = 25 \Rightarrow x = \pm\sqrt{25} = \pm 5$

ex: $x^2 = -16$ $\quad x = \pm\sqrt{-16} = \pm\sqrt{16}\sqrt{-1}$

$= \pm 4i$

That is: $x = 4i$ or $x = -4i$

Check: $(4i)^2 = 4^2 \cdot i^2 = 16(-1) = -16$

Check: $(-4i)^2 = (-4)^2(i^2) = 16(-1) = -16$

ex: $x^2 = 50 \Rightarrow x = \pm\sqrt{50}$
$= \pm\sqrt{25}\sqrt{2} = \pm 5\sqrt{2}$

That is $x = 5\sqrt{2}$ and $x = -5\sqrt{2}$ are both solutions.

check: $x = 5\sqrt{2}$
$(5\sqrt{2})^2 = 5^2 \cdot \sqrt{2}^2 = 25 \cdot 2 = 50$ ✓

Remark: This is our first quadratic equation which cannot be solved by factoring.

$x^2 = 50 \Rightarrow x^2 - 50 = 0$

doesn't factor (over the integers)

Principle of Square Roots: If $x^2 = k$
then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

ex: $(x-3)^2 = 49 \Rightarrow x - 3 = \pm\sqrt{49} = \pm 7$
$x - 3 = 7$ or $x - 3 = -7$
$\boxed{x = 10 \quad \text{or} \quad x = -4}$

ex: $(x+2)^2 = 5 \Rightarrow x + 2 = \pm\sqrt{5}$
$x = -2 \pm \sqrt{5}$

That is $x = -2 + \sqrt{5}$ or $x = -2 - \sqrt{5}$

ex: $(x-3)^2 = -16$ ← Given: $(\text{binomial})^2 = \text{constant}$ we're home free!!

$$x-3 = \pm\sqrt{-16} = \pm 4i$$

$$x-3 = \pm 4i$$

$$x = 3 \pm 4i$$ that is $3+4i$ or $3-4i$.

Claim: Any quadratic equation can be put in this form:

ex: $x^2 - 2x + 5 = 0$

$$x^2 - 2x = -5$$

$$x^2 - 2x + 1 = -5 + 1$$ ← $\frac{1}{2}(-2) = -1$, $(-1)^2 = 1$

$$(x-1)^2 = -4$$

$$x - 1 = \pm\sqrt{-4} = \pm 2i$$

$$x = 1 \pm 2i$$

That is $x = 1+2i$ is a solution, as is $x = 1-2i$

[See page 3 bottom]

Remark: This is our first example of solving by "completing the square."