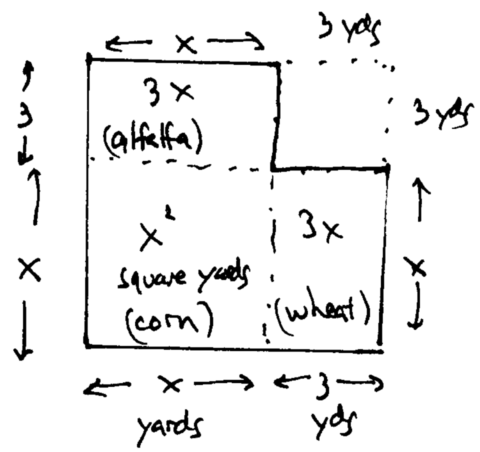
11.1 Completing the square (cont'd)

The idea of "completing the square"

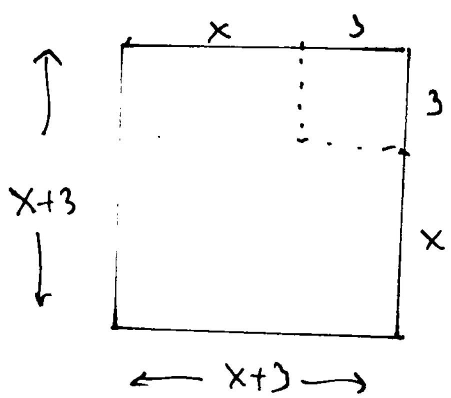
Suppose we are farmers with farm lots of area $x^2 + 6x$ square yards



Q: How much more land must we buy to complete the square (and have area to grow tomatos)?

A: 9 more square yards.

Q: After we buy the 9 square yards, what are the dimensions of our farm?



$$\underbrace{x^2+6x}_{\text{origial}} + \underbrace{9}_{\text{new}} = (x+3)^2$$

<u>more examples</u>

$$x^2 + 10x + \underline{25} = (x+5)^2$$

$$x^2 + 16x + 64 = (x+8)^2$$

$$x^2 - 2x + 1 = (x-1)^2$$

$$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$$

$$x^2 + 5x + \frac{25}{4} = \left(x + \frac{5}{2}\right)^2$$

i) $\frac{1}{2} \cdot 5 = \frac{5}{2}$

ii) $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$

<u>Reason</u>: Useful formulas

$(a+b)^2 = a^2 + 2ab + b^2$ or

$(a-b)^2 = a^2 - 2ab + b^2$

So $x^2 + 10x + \underline{25}$

$x^2 + 2 \cdot x \cdot 5 + 5^2$

$(x+5)^2$

want to resemble

$a^2 + 2ab + \_\_\_\_$

where $a = x$

$b = 5$

$$x^2 + \frac{11}{3}x + \frac{121}{36} = \left(x + \frac{11}{6}\right)^2$$

i) $\frac{1}{2}\left(\frac{11}{3}\right) = \frac{11}{6}$

ii) $\left(\frac{11}{6}\right)^2 = \frac{121}{36}$

To solve a quadratic eqn by completing the square

1. Get variables on one side, constant on other side.
2. Divide by the leading coefficient
3. Complete the square.
4. Factor the trinomial; simplify other side
5. Take square root of both sides
6. Solve for x.

ex: $3x^2 + 18x - 24 = 0$

(1) $3x^2 + 18x = 24$ ← Get constant on right

(2) $x^2 + 6x = 8$ ← Divide by leading coefficient

(3) $x^2 + 6x + 9 = 8 + 9$ ← Complete the square

(4) $(x+3)^2 = 17$ ← Factor left; simplify right

(5) $x + 3 = \pm\sqrt{17}$ ← square root principle

(6) $x = -3 \pm \sqrt{17}$ ← solve for x

ex: [Great situation for completing the square:
in $ax^2+bx+c=0$, where $a=1$ and b is even]

$$x^2-8x+20=0$$

① $x^2-8x = -20$

③ $x^2-8x+16 = -20+16$

④ $(x-4)^2 = -4$

⑤ $x-4 = \pm\sqrt{-4} = \pm 2i$

⑥ $x = 4 \pm 2i$

TRY THIS: Solve by completing the square.

ex: $x^2-10x+21 = 0$

① $x^2-10x = -21$

③ $x^2-10x+25 = -21+25$

④ $(x-5)^2 = 4$

⑤ $x-5 = \pm\sqrt{4} = \pm 2$

⑥ $x = 5 \pm 2 = \begin{cases} 7 & \text{or} \\ 3 \end{cases}$

← Hey wait a minute! we could have solved by factoring!

Begin again: $x^2-10x+21=0$

$(x-7)(x-3)=0$

$x-7=0$ or $x-3=0$

$x=7$ or $x=3$

Solve by completing the square

ex: $3x^2 + 11x + 5 = 0$

1. $3x^2 + 11x = -5$

2. $x^2 + \frac{11}{3}x = \frac{-5}{3}$

3. $x^2 + \frac{11}{3}x + \frac{121}{36} = -\frac{5}{3} + \frac{121}{36}$

4. $\left(x + \frac{11}{6}\right)^2 = \frac{-5 \cdot 12 + 121}{36}$

   $= \frac{61}{36}$

5. $x + \frac{11}{6} = \pm\sqrt{\frac{61}{36}} = \pm\frac{\sqrt{61}}{6}$

6. $x = -\frac{11}{6} \pm \frac{\sqrt{61}}{6} = \frac{-11 \pm \sqrt{61}}{6}$

scratch work

i) $\frac{1}{2}\left(\frac{11}{3}\right) = \frac{11}{6}$

ii) $\left(\frac{11}{6}\right)^2 = \frac{121}{36}$

11.2 Quadratic Formula

Let's derive the quadratic formula by imitating the previous example, but with "a" replacing 3, "b" replacing 11, "c" replacing 5.

$a = 3$
$b = 11$
$c = 5$

Solve by completing the square:

$$ax^2 + bx + c = 0$$

1 $$ax^2 + bx = -c$$

2 $$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3 $$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

Scratch

i) $\frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a}$

ii) $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$

4 $$\left(x + \frac{b}{2a}\right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

5 $$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

6 $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic formula: The solutions to $ax^2 + bx + c = 0$ are:

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

← Memorize this!

example: Solve $3x^2 + 11x + 5 = 0$ using this formula:

$$x = \frac{-(11) \pm \sqrt{(11)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{-11 \pm \sqrt{121 - 60}}{6} = \frac{-11 \pm \sqrt{61}}{6}$$

Solve by the quadratic formula

ex: $2x^2 - 5x + 7 = 0$ ← $a = 2$, $b = -5$, $c = 7$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(7)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25 - 56}}{4}$$

$$= \frac{5 \pm \sqrt{-31}}{4} = \frac{5 \pm i\sqrt{31}}{4}$$

ex: $3x^2 + 10x + 8 = 0$

$$x = \frac{-(10) \pm \sqrt{(10)^2 - 4(3)(8)}}{2(3)}$$

$$= \frac{-10 \pm \sqrt{100 - 96}}{6} = \frac{-10 \pm \sqrt{4}}{6}$$

$$= \frac{-10 \pm 2}{6} = \frac{\cancel{2}(-5 \pm 1)}{\cancel{2} \cdot 3} = \frac{-5 \pm 1}{3} = \begin{cases} -\frac{4}{3} & \text{OR} \\ -2 \end{cases}$$

Hey! ~~We~~ could have solved by factoring! (Because there are no radicals in the answer.)

$3x^2 + 10x + 8 = 0$

$(x+2)(3x+4) = 0$

$x+2 = 0$ OR $3x+4 = 0$

$x = -2$ OR $3x = -4$

$x = -\frac{4}{3}$

ex: Solve by the quadratic formula

$$x(x+2) = -5$$

what are a, b and c?
I dunno. Multiply out.

$$x^2+2x = -5$$

$$x^2+2x+5 = 0$$

So $a=1$, $b=2$, $c=5$

$$x = \frac{-2 \pm \sqrt{2^2-4(1)(5)}}{2(1)} = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = \frac{-2}{2} \pm \frac{4i}{2} = -1 \pm 2i$$

solutions: $x = -1+2i$ OR $x = -1-2i$

check: $x = -1+2i$ is a solution. Note: $x+2 = +1+2i$

$$x(x+2) = (-1+2i)(1+2i) = -1 - \cancel{2i} + \cancel{2i} + 4i^2$$

$$= -1 - 4 = -5 \checkmark$$

ex: $\frac{1}{2}x^2 + \frac{5}{3}x - \frac{7}{6} = 0$

$a = 1/2$
$b = 5/3$
$c = -7/6$

Let's multiply both sides by 6:

$$\frac{6}{2}x^2 + \frac{6\cdot 5}{3}x - \frac{6\cdot 7}{6} = 6\cdot 0$$

$$3x^2 + 10x - 7 = 0$$

Now $a=3$
$b=10$
$c=-7$

$$x = \frac{-10 \pm \sqrt{10^2-4(3)(-7)}}{2(3)} = \frac{-10 \pm \sqrt{100+84}}{6}$$

$$= \frac{-10 \pm \sqrt{184}}{6} = \frac{-10 \pm 2\sqrt{46}}{6} = \frac{2(-5 \pm \sqrt{46})}{6} = \frac{-5 \pm \sqrt{46}}{3}$$