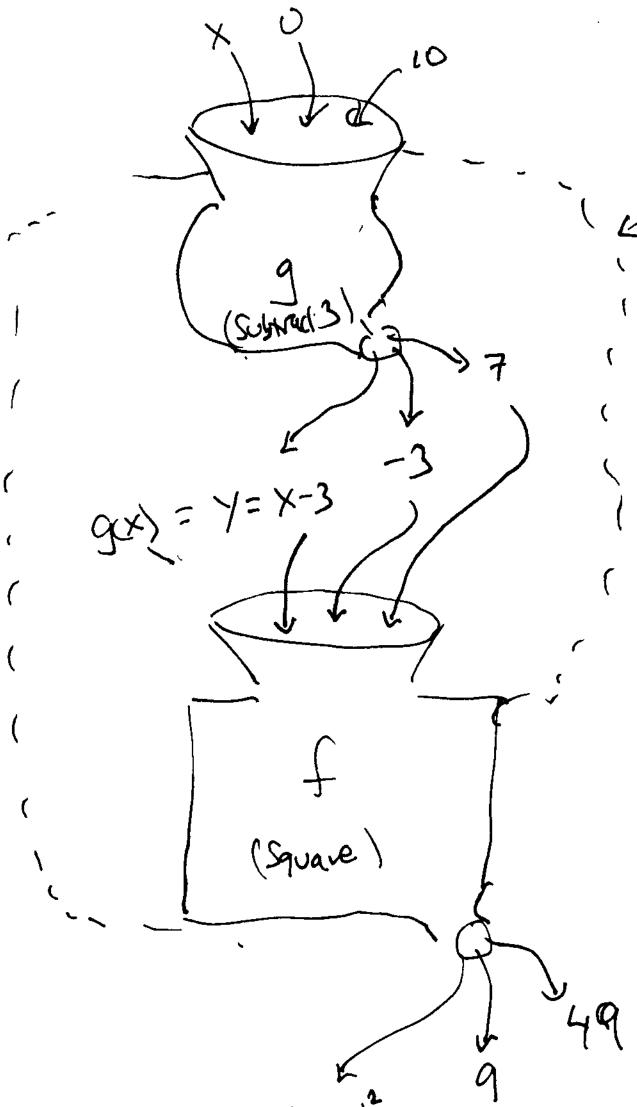
12.1 Composition of functions

<u>Idea</u>: If $y = g(x) = x - 3$

and $z = f(y) = y^2$



"composition of f and g" denoted "$f \circ g$"

By definition:

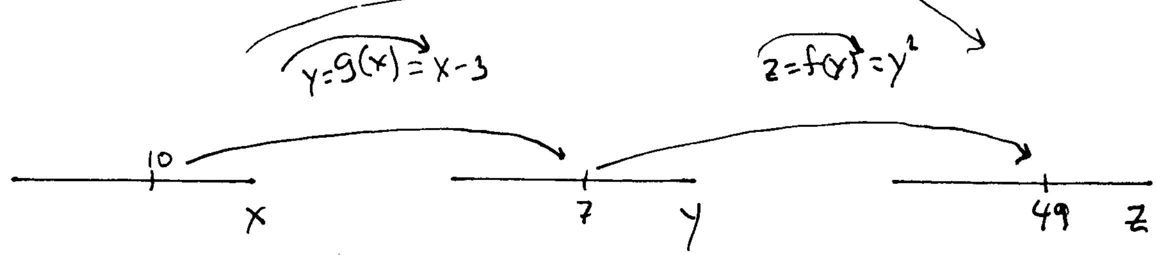
$$(f \circ g)(x) = f(g(x))$$

In our example,

$$(f \circ g)(10) = f(g(10))$$
$$= f(10-3)$$
$$= f(7)$$
$$= 7^2$$
$$= 49$$

$$f(g(x)) = (x-3)^2$$
$$= x^2 - 6x + 9$$

$z = f(g(x)) = (x-3)^2 = x^2 - 6x + 9$



<u>example</u> of composition in real life

x = cash (dollars)

$y = g(x)$ = gallons ~~of~~ gas you can buy with x dollars.

$= \frac{1}{3} x$

so if you have $x = 30$ dollars you buy

$y = g(30) = \frac{1}{3}(30) = 10$ gallons of gas.

$z = f(y)$ = miles you can drive with y gallons of gas

$= 20y$ (assuming you get 20 mpg)

We can interpret

$z = (f \circ g)(x) = f(g(x))$ = number of miles of travel you get from x dollars

e.g. If you have 30 dollars,

then $y = g(30) = \frac{1}{3}(30) = 10$ gallons

and $z = f(10) = 20(10) = 200$ miles

More generally, $f(g(x)) = f\left(\frac{1}{3}x\right) = 20\left(\frac{1}{3}x\right) = \frac{20}{3}x$

$\approx 6.66x$ miles

M96 11/12/15

13) $f(x) = x+7 \qquad g(x) = \frac{1}{x^2}$

Find a) $(f \circ g)(1)$

b) $(g \circ f)(1)$

c) $(f \circ g)(x)$

d) $(g \circ f)(x)$

← evaluate inside to outside

a) $(f \circ g)(1) = f(g(1)) = f\left(\frac{1}{1^2}\right)$

$= f(1) = 1+7 = 8$

b) $(g \circ f)(1) = \cancel{f(g(1)) = f(1+7) = f(8)}$

$= g(f(1)) = g(1+7) = g(8)$

$= \frac{1}{8^2} = \frac{1}{64}$ Note: $8 \neq \frac{1}{64}$

c) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x^2}\right) = \frac{1}{x^2} + 7$

d) $(g \circ f)(x) = g(f(x)) = g(x+7) = \frac{1}{(x+7)^2}$

TRY THIS! 9) $f(x) = x^2+1$ $\qquad g(x) = x-3$

Find a) $(f\circ g)(1)$ b) $(g\circ f)(1)$ c) $(f\circ g)(x)$ d) $(g\circ f)x$.

a) $f(g(1)) = f(1-3) = f(-2) = (-2)^2+1 = \boxed{5}$

b) $g(f(1)) = g(1^2+1) = g(2) = 2-3 = \boxed{-1}$

c) $f(g(x)) = f(x-3) = \boxed{(x-3)^2+1} = x^2-6x+9+1$
$= \boxed{x^2-6x+10}$

d) $g(f(x)) = g(x^2+1) = (x^2+1)-3 = \boxed{x^2-2}$

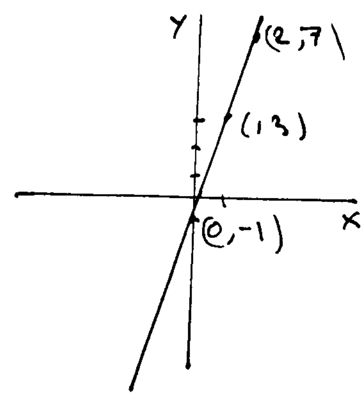
One-to-one functions

Defn: A function f is one-to-one if
$f(a) \neq f(b)$ whenever $a \neq b$.

ex: $f(x) = 4x-1$ is a one-to-one function

x	$f(x)$
-1	-5
0	-1
1	3
2	7
3	10

No repeated numbers in 2nd column

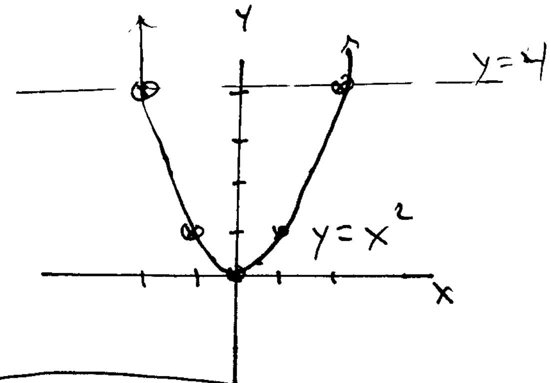


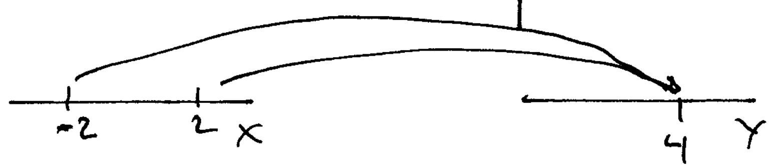
← The graph passes the horizontal line test: Any horizontal line crosses the graph only once.

ex (not one-to-one)

$$g(x) = x^2$$

x	x^2
-2	4
-1	1
0	0
1	1
2	4





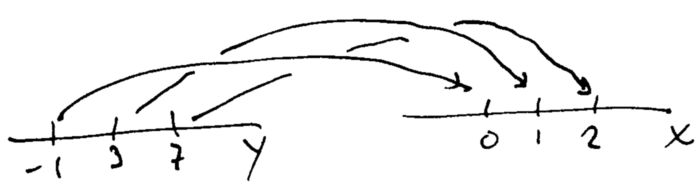
Inverse functions

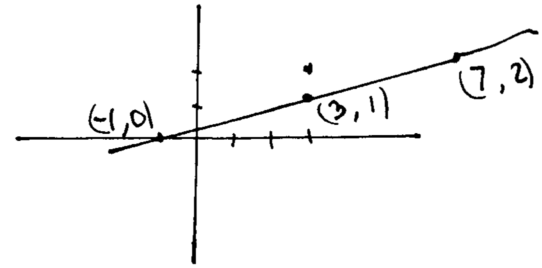
Informal Definition: If f is a one-to-one function, we can define a new function, called the inverse of f, by swapping the role of input and output.

Defn: Suppose f is a one-to-one function. Then, there is another function, denoted f^{-1} (lousy notation), defined by $y = f(x)$ means $f^{-1}(y) = x$.

ex: $f(x) = 4x - 1$ what is $f^{-1}(y)$?

y	$f^{-1}(y)$
-5	-1
-1	0
3	1
7	2
10	3





ex (cont'd): Now, what is the equation of f^{-1}?

Start with $f(x) = 4x - 1$

Rewrite as $y = 4x - 1$

(key step) Swap x and y $\quad x = 4y - 1$

Solve for y: $\quad x+1 = 4y - 1 + 1$

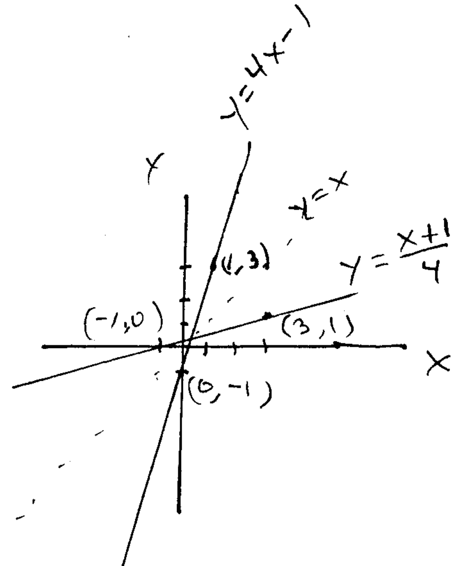
(do the same thing on both sides of the equal sign) $\quad x+1 = 4y$

$$\frac{x+1}{4} = \frac{4y}{4}$$

$$\frac{x+1}{4} = y$$

Answer: $\boxed{f^{-1}(x) = \frac{x+1}{4}}$

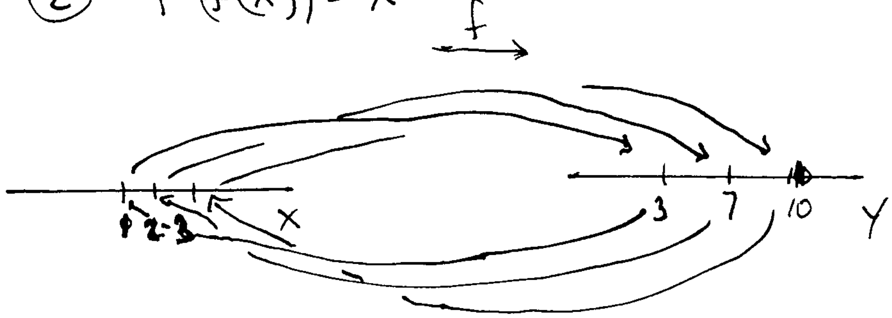
$= \frac{1}{4}x + \frac{1}{4}$



FACT: If f and f^{-1} are inverses of each other, then

(1) $f(f^{-1}(x)) = x$

(2) $f^{-1}(f(x)) = x$



(1) $f(f^{-1}(7)) = f(2) = 7$

(2) $f^{-1}(f(2)) = f^{-1}(7) = 2$