Math 96 Thu 19 Nov 2015

12.4 Properties of logs AND 12.5 Common logs and natural logs

Six Properties of Logs

1. $\log_a a^x = x$ } Inverse properties
2. $a^{\log_a x} = x$

3. $\log_a (MN) = \log_a M + \log_a N$ Product Rule for logs
4. $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$ Quotient Rule "
5. $\log_a (M^p) = p \cdot \log_a M$ Power Rule "
6. $\log_b M = \dfrac{\log_a M}{\log_a b}$ Change-of-base formula

ex of 1.

$\log_2 2^5 = 5$ $\log_{10} 10^{1/2} = \frac{1}{2}$

$\log_5 5^3 = 3$ $\log_6 6^1 = 1$

$\log_9 9^0 = 0$

explanation: $(2^5) = 2^5$ is equivalent to

$\underset{\text{other}}{(2^5)} = \underset{\text{base}}{2}^{5 \leftarrow \text{exp}}$

$\log_2 (2^5) = 5$

Remark. This allows for shortcuts in evaluating logs.

ex: $\log_5 125 = \log_5 5^3 = 3$

ex: $\log_7 \frac{1}{49} = \log_7 7^{-2} = -2$

ex: $\log_{15} 15 = \log_{15} 15^1 = 1$

ex: $\log_8 1 = \log_8 8^0 = 0$

ex: $\log_2 7$ Hmmm. I think we're stuck. We need a calculator.

examples of (2):

$10^{\log_{10} 13} = 13$

$2^{\log_2 8} = 8$

$e^{\log_e 21} = 21$ $e = 2.718281828$ is a constant

explanation:

$\underset{\substack{\uparrow \\ \text{base}}}{\log_{10}} 13^{\text{other}} = (\log_{10} 13) \leftarrow$ exponent

Equivalent exponential equation: $10^{\log_{10} 13} = 13$

NOTE: $\log_{10} 1{,}000{,}000 = 6$

why? $\log_{10} 1{,}000{,}000 = \log_{10} 10^6 = 6$

Also $\log_{10} 100 = 2$. More generally $\log_{10} 10^n = n$.

Useful for coming up with simple examples.

exaple of 3 (product rule)

$$\log_{10}[(100)(1000)] = \log_{10} 100 + \log_{10} 1000$$
$$5 = 2 + 3$$

In words: "The log of a product is the sum of the logs."

exaple of 4 (quotient rule)

$$\log_{10}\left(\frac{100{,}000}{1000}\right) = \log_{10} 100{,}000 - \log_{10} 1000$$
$$2 = 5 - 3$$

In words: "The log of a quotient is the difference of the logs."

Exaple of 5 (power rule)

$$\log_{10}(100^3) = 3 \cdot \log_{10} 100$$
$$6 = 3 \cdot 2$$

In words: "The log of a power is the multiple of the logs."

Remark: we could have, instead, used the power rule twice.

$$\log_{10}(100^3) = \log_{10}(100 \cdot 100 \cdot 100)$$
$$= \log_{10} 100 + \log_{10} 100 + \log_{10} 100$$
$$= 2 + 2 + 2 = 3 \cdot 2 = 3 \log_{10} 100$$

12.4 13) write as a single log.

$$\log_a 2 + \log_a 10 = \log_a(2\cdot 10) = \log_a 20$$

14) $\log_b 5 + \log_b 9 = \log_b(5\cdot 9) = \log_b 45$

30) $\log_a 26 - \log_a 2 = \log_a\left(\frac{26}{2}\right) = \log_a 13$

write as a product:

19) $\log_2 y^{1/3} = \frac{1}{3}\log_2 y$

ex ["Expanding log expressions"]

36) $\log_a(x^2y^5) = \log_a x^2 + \log_a y^5$ (product)

$= 2\log_a x + 5\log_a y$ (power rule)

42) $\log_b \frac{w^2x}{y^3z} = \log_b w^2x - \log_b y^3z$ (quotient)

$= (\log_b w^2 + \log_b x) - (\log_b y^3 + \log_b z)$ (by quotient rule)

$= \log_b w^2 + \log_b x - \log_b y^3 - \log_b z$

$= 2\log_b w + \log_b x - 3\log_b y - \log_b z$

Alternative method: $\log_b\left(\frac{w^2x}{y^3z}\right) = \log_b(w^2xy^{-3}z^{-1})$

$= \log_b w^2 + \log_b x + \log_b y^{-3} + \log_b z^{-1}$

$= 2\log_b w + \log_b x - 3\log_b y - \log_b z$

ex. ["Combine into a single log"]

49) $8 \log_a x + 3 \log_a z = \log_a x^8 + \log_a z^3$ (power)

$= \log_a (x^8 \cdot z^3)$ (product rule)

52) $3 \log_c p + \frac{1}{2} \log_c t - \log_c 7$

$= \log_c p^3 + \log_c t^{1/2} - \log_c 7$

$= \log_c p^3 t^{1/2} - \log_c 7 = \log_c \left(\frac{p^3 t^{1/2}}{7}\right) = \log_c\left(\frac{p^3\sqrt{t}}{7}\right)$

57) $\log_a (x^2-9) - \log_a (x+3)$

$= \log_a \left(\frac{x^2-9}{x+3}\right)$ (by the Quotient Rule backwards)

$= \log_a \left(\frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}}\right) = \log_a (x-3)$

Idea behind the change-of-base formula: (Need a calculator)

ex: Find $\log_2 16$. Easy... $\log_2 16 = \log_2 2^4 = 4$.
So we got lucky.

ex: Find $\log_2 14$. Let $x = \log_2 14$

(*) The trick → Equivalent: $2^x = 14$ Now* take $\log_{10}$ of both sides

$\log_{10} 2^x = \log_{10} 14 \Rightarrow x \log_{10} 2 = \log_{10} 14$ (by power rule)

Divide both sides of eqn by $\log_{10} 2$.

$\therefore x = \frac{\log_{10} 14}{\log_{10} 2} = \frac{1.146128036}{0.301029957} = 3.807354922$

Note. It little less than 4.

Derivation of the change-of-base formula:

(6) $$\boxed{\log_b M = \frac{\log_a M}{\log_a b}}$$

Reason: Find $\log_b M$. Let $x = \log_b M$

Equivalent: $b^x = M$

Take $\log_a$ of both sides:

$$\log_a b^x = \log_a M$$

Use the power rule:

$$x \cdot \log_a b = \log_a M$$

Divide both sides by $\log_a b$: $$x = \frac{\log_a M}{\log_a b}$$

ex: Use a calculator to find $\log_7 25$, by letting $M=25$, $b=7$, $a=10$:

$$\log_7 25 = \frac{\log_{10} 25}{\log_{10} 7} = \frac{1.398}{0.845} = 1.654\ldots$$

ex: without a calculator, find exactly

(So cool)

$$\log_8 16 = \frac{\log_{10} 16}{\log_{10} 8} = \frac{\log 2^4}{\log_{10} 2^3} = \frac{4 \cdot \cancel{\log_{10} 2}}{3 \cdot \cancel{\log_{10} 2}}$$

$$= \boxed{\frac{4}{3}}$$