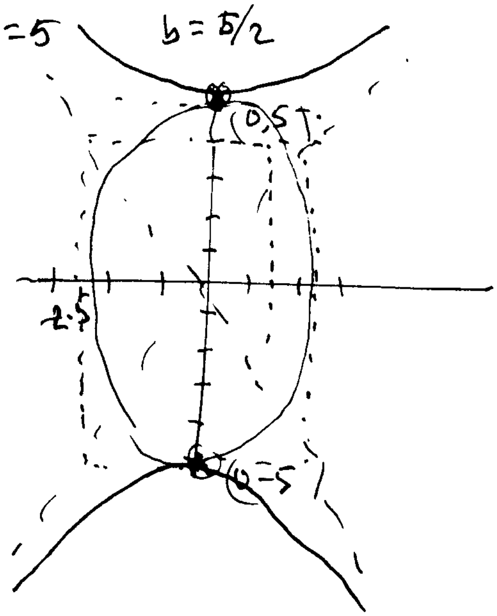
13.4 One more example of a nonlinear system

30) (1) $\begin{cases} y^2 - 4x^2 = 25 \\ 4x^2 + y^2 = 25 \end{cases}$ (2)

(1) is equivalent to

hyperbola $\dfrac{y^2}{25} - \dfrac{x^2}{(25/4)} = 1$

$a = 5$ $b = 5/2$



(2) ellipse

$\dfrac{x^2}{(25/4)} + \dfrac{y^2}{25} = 1$

$b^2 = \dfrac{25}{4}$ $a^2 = 25$

$a = 5$

$b = \dfrac{5}{2}$

30 cont'd) Let's try to solve by elimination the system

(1) $-4x^2 + y^2 = 25$

(2) $4x^2 + y^2 = 25$

$$2y^2 = 50 \qquad \text{so } x^2 \text{ is eliminated}$$

$$\frac{2y^2}{2} = \frac{50}{2}$$

$$y^2 = 25 \Rightarrow y = \pm\sqrt{25} = \pm 5$$

Back substitute: Use (2) to find x:

$$4x^2 + 25 = 25$$

$$4x^2 = 0$$

$$x^2 = 0 \Rightarrow x = 0$$

Two solutions: $(x,y) = (0,5)$ or $(x,y) = (0,-5)$

14.1 Sequences and series

Informal definition: A sequence is an infinite list of numbers.

examples:

1. $5, 9, 13, 17, 21, 25, \ldots$ "arithmetic sequence"
2. $3, 6, 12, 24, 48, 96, \ldots$ "geometric sequence"
3. $1, 4, 9, 16, 25, 36, \ldots$ neither

Formal definition: A sequence is a function whose domain is the set of positive integers.

input $= n =$ where we are in the list

output $= a_n =$ the number at this location in the list.

example 3 again [Notation] $\boxed{a_n = n^2}$ for $n = 1, 2, 3, \ldots$

"Explicit definition of a_n"

What is a_{10}? $a_{10} = 10^2 = 100$

what is a_7? $a_7 = 7^2 = 49$

For what n is $a_n = 121$? $n = 11$.

16) $a_n = (3n+2)^2$ what is a_6? $a_6 = (3 \cdot 6 + 2)^2 = 20^2 = 400$

write the first six terms

n	1	2	3	4	5	6
a_n	25	64	121	196	289	400

28) $$a_n = \frac{n^2-1}{n^2+1}$$

First four terms: $a_1 = \frac{1^2-1}{1^2+1} = 0$

$$a_2 = \frac{2^2-1}{2^2+1} = \frac{3}{5}$$

$$a_3 = \frac{3^2-1}{3^2+1} = \frac{8}{10}$$

$$a_4 = \frac{4^2-1}{4^2+1} = \frac{15}{17}$$

$$\vdots$$

$$a_{10} = \frac{10^2-1}{10^2+1} = \frac{99}{101}$$

$$\vdots$$

$$a_{15} = \frac{15^2-1}{15^2+1} = \frac{224}{226}$$

<u>Defn</u>: Given a sequence $a_1, a_2, a_3, a_4, \ldots$

the expression $a_1 + a_2 + a_3 + a_4 + \cdots$ is an <u>infinite series</u>

The "n^{th} partial sum" S_n

is $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

<u>ex</u>: Given the sequence $32, 16, 8, 4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

$$S_1 = 32$$
$$S_2 = 32 + 16$$
$$S_3 = 32 + 16 + 8$$
$$S_4 = 32 + 16 + 8 + 4$$
$$\vdots$$
$$S_\infty = 32 + 16 + 8 + 4 + 2 + \cdots$$

Sigma notation

final value, typical term

ex: $$\sum_{k=1}^{5} (2k+1)$$

index, initial value

$$= (2\cdot1+1) + (2\cdot2+1) + (2\cdot3+1) + (2\cdot4+1) + (2\cdot5+1)$$

$$= 3 + 5 + 7 + 9 + 11$$

Why use sigma notation?

(1) To save space. But more importantly..

(2) To express a pattern.

65) Write out and evaluate

$$\sum_{k=3}^{5} \frac{(-1)^k}{k(k+1)} = \frac{(-1)^3}{3(3+1)} + \frac{(-1)^4}{4(4+1)} + \frac{(-1)^5}{5(5+1)}$$

$$= -\frac{1}{3\cdot4} + \frac{1}{4\cdot5} - \frac{1}{5\cdot6}$$

$$= \frac{-1}{12} + \frac{1}{20} - \frac{1}{30}$$

$$= -\frac{5}{60} + \frac{3}{60} - \frac{2}{60}$$

$$= \frac{-5+3-2}{60} = \frac{-4}{60} = -\frac{1}{15}$$

14.2 Arithmetic sequences

Defn: A sequence is arithmetic if it is recursively defined by

$$\boxed{a_{n+1} = a_n + d}$$

for some fixed number d, where d is called the common difference.

example: $\begin{cases} a_1 = 4 \\ a_{n+1} = a_n + 5 \end{cases}$ ← so $d=5$

$$a_2 = a_{1+1} = a_1 + 5 = 4+5 = 9$$
$$a_3 = a_{2+1} = a_2 + 5 = 9+5 = 14$$
$$a_4 = a_{3+1} = a_3 + 5 = 14+5 = 19$$

ex: 27, 20, 13, 6, −1, −8, ...

Is this an arithmetic sequence?

What is a_1? $a_1 = 27$

What is d? $d = -7$

ex: 3, 3.5, 4, 4.5, 5, 5.5.

$a_1 = 3$

$d = 0.5$

Remark: The problem with recursively defined sequence: What is a_{1000}?

ex: 4, 9, 14, 19, 24, 29, ..

so $a_1 = 4$, $d = 5$.

what is a_{11}?

$$4 \xrightarrow{+5} 9 \xrightarrow{+5} 14 \xrightarrow{+5} 19, \ldots \xrightarrow{+5}$$

$$\underset{a_1}{4},\ \underset{a_2}{9},\ \underset{a_3}{14},\ \underset{a_4}{19},\ \ldots\ \underset{a_{11}}{\uparrow}$$

$$a_{11} = \underset{a_1}{4} + \underset{\substack{\text{number of} \\ \text{"jumps"} \\ =11-1}}{10} \cdot \underset{d}{5} = 4 + (11-1)\cdot 5 = 54$$

what about a_{101}?

$$a_{101} = 4 + 100 \cdot 5 = 504$$

$$\boxed{a_n = a_1 + (n-1)\cdot d}$$

n^{th} term in an arithmetic sequence

(6) Find the 14th term in the arithmetic sequence

$$3,\ \frac{7}{3},\ \frac{5}{3},\ \ldots$$

$a_1 = 3$, $d = -\frac{2}{3}$, $n = 14$, $a_n = a_{14} = ?$

$$a_{14} = 3 + (14-1)\left(-\frac{2}{3}\right) = 3 + (13)\left(-\frac{2}{3}\right)$$

$$= 3 - \frac{26}{3} = \frac{9}{3} - \frac{26}{3} = -\frac{17}{3}$$

22) In ~~problem~~ 16 what term is −27 ?

That is, $a_n = -27$ for what n?

Know: $a_1 = 3$, $d = -\frac{2}{3}$, $a_n = -27$, $n = ?$

$$-27 = 3 + (n-1)\left(-\frac{2}{3}\right)$$ ← solve for n

$$-3 \qquad -3$$

$$-30 = (n-1)\cdot\left(-\frac{2}{3}\right)$$

multiply by $-\frac{3}{2}$:

$$(-30)\cdot\left(\frac{-3}{2}\right) = (n-1)\cdot\left(-\frac{2}{3}\right)\cdot\left(-\frac{3}{2}\right)$$

$$\frac{90}{2} = n-1$$

$$45 = n-1$$

$$+1 \qquad +1$$

$$46 = n$$

Sum of first n terms of an arithmetic sequence

ex: 5, 9, 13, 17, 21 ↙ what's the sum?

$$S_5 = 5 + 9 + 13 + 17 + 21$$

Trick:

$$S_5 = 21 + 17 + 13 + 9 + 5$$

$$2\,S_5 = 26 + 26 + 26 + 26 + 26 = 5 \cdot (26)$$

$$= 5(5 + 21)$$

$$S_5 = \frac{5(5+21)}{2}$$

$$\boxed{S_n = \frac{n(a_1 + a_n)}{2}}$$

Sum of the first n terms of an arithmetic sequence

$$= n \cdot \frac{(a_1 + a_n)}{2}$$

↓ Added after class ended.

39) Find the sum of all multiples of 6 from 6 to 102.

That is, find $S_n = 6 + 12 + 18 + \cdots + 102$.

Known: $a_1 = 6$, $d = 6$; $a_n = 102$. Not known: n

Use $a_n = a_1 + (n-1) \cdot d$ to get $102 = 6 + (n-1) \cdot 6 = 6n$

so that $n = \frac{102}{6} = 17$.

Then $S_n = S_{17} = \frac{n \cdot (a_1 + a_n)}{2} = \frac{17 \cdot (6 + 102)}{2} = \boxed{918}$