14.2

37) Find the sum of the even numbers from 2 to 100 inclusive:

$$S_n = 2 + 4 + 6 + 8 + \cdots + 96 + 98 + 100$$

Notice: $2, 4, 6, 8, \ldots, 100, \ldots$

is an arithmetic sequence with $\begin{cases} a_1 = 2 \\ d = 2 \end{cases}$

we want to use $\boxed{S_n = \dfrac{n(a_1 + a_n)}{2}}$ with $a_1 = 2$, $a_n = 100$.

But what would n equal?

To find n, let's use $\boxed{a_n = a_1 + (n-1)d}$ with $\begin{cases} a_1 = 2 \\ d = 2 \\ a_n = 100 \end{cases}$

$$100 = 2 + (n-1)\cdot 2$$
$$100 = 2 + 2n - 2$$
$$100 = 2n \Rightarrow n = 50$$

Back to S_n:

$$S_{50} = S_n = \frac{n(a_1 + a_n)}{2} = \frac{50(2+100)}{2} = \frac{50(102)}{2}$$
$$= 50(51) = 2{,}550$$

14.3 Geometric sequences

ex ① $3, -6, 12, -24, 48, \ldots$ $\begin{cases} a_1 = 3 \\ r = -2 \end{cases}$

ex ②: $.45, \ .0045, \ .000045, \ \ldots$

$\begin{cases} a_1 = .45 \\ r = .01 = \frac{1}{100} \end{cases}$

ex Find the 11^{th} term in example ①.

$3 \xrightarrow{\cdot(-2)} -6 \xrightarrow{\cdot(-2)} 12 \xrightarrow{\cdot(-2)} -24, \ldots \xrightarrow{\cdot(-2)} 3072$

a_1 ... a_{11}

So $a_{11} = 3 \cdot \underbrace{(-2)(-2)(-2) \cdots (-2)}_{10 = 11-1}$

$= 3 \cdot (-2)^{11-1} = 3 \cdot (-2)^{10} = 3\,(1024)$
$= 3072$

$\boxed{a_n = a_1 \cdot r^{n-1}}$ n^{th} term of a geometric series

ex: Find the 10^{th} term of the geometric sequence

$64, -32, 16, -8, \ldots$ $a_1 = 64$ $r = \frac{-1}{2}$

So $a_{10} = a_1 \cdot r^{n-1} = 64 \cdot \left(-\frac{1}{2}\right)^9 = \frac{-64}{512} = \frac{-2^6}{2^9} = \frac{-1}{2^3}$

Answer: $-\frac{1}{2^3} = \frac{-1}{8}$

Sum of the first n terms of a geometric sequence

$\cancel{S_n = a}$ $S_1 = a_1 = a_1$

$S_2 = a_1 + a_2 = a_1 + a_1 r$

$S_3 = a_1 + a_2 + a_3 = a_1 + a_1 r + a_1 r^2$

$\vdots$

$$S_n = a_1 + a_1 r + a_1 r^2 + \dots + \underbrace{a_1 r^{n-1}}_{a_n}$$

Trick: $r S_n = \quad a_1 r + a_1 r^2 + \dots + a_1 r^{n-1} + a_1 r^n$

Subtract: $S_n - r S_n = a_1 + 0 + 0 + \dots + 0 - a_1 r^n$

$(1-r) S_n = a_1 (1 - r^n)$ so

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

example: Find the sum of the first 9 terms of

$-5, 10, -20, 40, \dots$

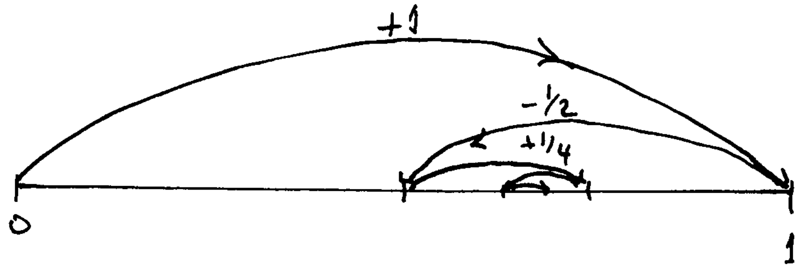
so $a_1 = -5$, $r = -2$, $n = 9$

$$S_9 = \frac{-5\,(1-(-2)^9)}{1-(-2)} = \frac{(-5)[1-(-512)]}{3}$$

$$= \frac{(-5)(513)}{3} = -855$$

ex: A flea is jumping on a meter stick.

It jumps forward 1 meter, then backward $\frac{1}{2}$ meter, the forward $\frac{1}{4}$ meter etc.



Where is the flea after 5 jumps? after infinitely many jumps?

After 5 jumps: $1 + \left(-\frac{1}{2}\right) + \left(\frac{1}{4}\right) + \left(-\frac{1}{8}\right) + \left(\frac{1}{16}\right) = S_5$

Note: This is the sum of the first 5 terms of geometric sequence where

$1,\ -\frac{1}{2},\ \frac{1}{4},\ \ldots$ $\quad \begin{cases} a_1 = 1 \\ r = -\frac{1}{2} \end{cases}$

$$S_5 = a_1 \cdot \frac{(1-r^n)}{1-r} = 1 \cdot \frac{1-\left(-\frac{1}{2}\right)^5}{1-\left(-\frac{1}{2}\right)} = \frac{1+\frac{1}{32}}{\frac{3}{2}}$$

$$= \frac{\frac{33}{32}}{\frac{3}{2}} = \frac{33}{32} \div \frac{3}{2} = \frac{33}{32} \cdot \frac{2}{3} = \boxed{\frac{11}{16}}$$

Idea: when n is huge $\left(-\frac{1}{2}\right)^{\text{huge}} \approx 0$

$$S_\infty = a_1 \frac{1 - \cancel{r^n}^{\,0}}{1-r} = a_1 \cdot \frac{1}{1-r} = 1 \cdot \frac{1}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{1}{\frac{3}{2}} = \boxed{\frac{2}{3}}$$

Infinite geometric series

If $|r|<1$ [like $-\frac{1}{2}$]

$$\boxed{S_\infty = \frac{a_1}{1-r}}$$

example: $0.45454545\,45\,4\overline{5}$ represents what fraction?

$0.45 + 0.0045 + 0.000045 + \ldots$

is an infinite geometric series with $a_1 = 0.45$ and $r = 0.01$

$$S_\infty = \frac{a_1}{1-r} = \frac{.45}{1-.01} = \frac{.45}{.99} = \frac{45}{99}$$

$$= \boxed{\frac{5}{11}}$$

Check:

$$\begin{array}{r} .454545 \\ 11\overline{)5.000000} \\ \underline{44} \\ 60 \\ \underline{55} \\ 50 \end{array}$$