## §1.2 Gaussian elimination (loose ends)

### Homogeneous systems

zeros here

ex:
$$\begin{aligned} 2x + 4y - 7z &= 0 \\ x - 3y + 9z &= 0 \end{aligned}$$

NOTE: Every homogeneous system is consistent, for $(x,y,z)=(0,0,0)$, by inspection, a solution.
Are there any other solutions.

$$\left[\begin{array}{ccc|c} 2 & 4 & -7 & 0 \\ 1 & -3 & 9 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_2$
$$\left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 2 & 4 & -7 & 0 \end{array}\right]$$

$R_2 + (-2)R_1 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 0 & 10 & -25 & 0 \end{array}\right]$$

$\frac{1}{10}R_2 \to R_2$
$$\left[\begin{array}{ccc|c} 1 & -3 & 9 & 0 \\ 0 & 1 & -2.5 & 0 \end{array}\right] \leftarrow \text{ref}$$

$R_1 + 3R_2 \to R_1$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1.5 & 0 \\ 0 & 1 & -2.5 & 0 \end{array}\right] \leftarrow \text{rref}$$

NOTE: For homogeneous systems, we don't need to write the augmented matrix; we can write the coefficient matrix.

Let $z = t$
$$\begin{cases} x + 1.5z = 0 \\ y - 2.5z = 0 \end{cases}$$

Solution
$$\begin{cases} x = -1.5t \\ y = 2.5t \\ z = t \end{cases}$$
$\leftarrow$ Geometrically: A parametric line through the origin.

## Row equivalence

<u>Defn</u>: Two matrices are <u>row</u>-<u>equivalent</u> if they are the same size and there is a finite number of elementary row operations to get from the first matrix to the second matrix.

<u>Note</u>: This is an "equivalence relation", that is:

1. a matrix $A$ is row equivalent to itself (the <u>number</u> of row ops required is zero); so the relation is <u>reflexive</u>.

2. If matrix $A$ is row equivalent to matrix $B$, then matrix is row equivalent to matrix $A$.

"Row equivalence is <u>symmetric</u>"

   <u>Reason</u>: Each of the three row ops. are invertible and the inverse is itself an elementary row operation.

   <u>ex</u>: $R_1 \leftrightarrow R_3$ can be undone by another $R_1 \leftrightarrow R_3$

   $10R_2 \to R_2$ can be undone by $\frac{1}{10}R_2 \to R_2$.

   $R_2 + 3R_1 \to R_2$ can be undone by $R_2 + (-3)R_1 \to R_2$

   <u>Note</u>: To undo (say) five row ops, you would undo each of them, in reverse order.

3. If $A$ is row equivalent to $B$ and $B$ is row equivalent to a matrix $C$, then, by concatenating ~~both~~ sets of elementary row ops, we see that $A$ ~~is row~~ is row equivalent to $C$. So row equivalence is <u>transitive</u>.

Fact: While row echelon forms are not unique for a given matrix $A$, the reduced row echelon form of $A$ is unique, so we may regard the reduced row echelon form of a matrix to be a canonical element representing each (row) equivalence class.

ex: $A = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Are $A$ and $B$ row equivalent?

Let's find the rref of each.

Take $B$ and put it into rref:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad R_2 + (-1)R_1 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$R_1 + (-1)R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} = \mathrm{rref}(B)$$

Likewise with $A$:

$$\begin{bmatrix} 2 & 4 & 8 \\ 3 & 5 & 9 \end{bmatrix} \quad \tfrac{1}{2}R_1 \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 9 \end{bmatrix}$$

$$R_2 + (-3)R_1 \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \end{bmatrix}$$

$$-1R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix} \quad R_1 + (-2)R_2 \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} = \mathrm{rref}(A)$$

YES! $A$ and $B$ are row equivalent.

ex: Write every possible $2\times 3$ reduced-row echelon form.

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

a = any number, possibly zero.
b = "

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Remark</u>: Note that there are not just 7 row equivalence classes of $2\times 3$ matrices. Rather, there are <u>infinitely</u> <u>many</u> equivalence classes, one for each possible a and b. For instance,

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

each are members of <u>distinct</u> (row) equivalence classes.

## 1.3 Application: polynomial interpolation

2) Find a parabola (with vertical axis) passing through

passing through $(0,0)$ $(2,-2)$ $(4,0)$

<u>Idea</u>: A general quadratic function $p$ has the form:
$$p(x) = ax^2 + bx + c$$

$p(0)=0$: $\quad c = 0$

$p(2)=-2$: $\quad a\cdot 2^2 + b\cdot 2 + c = -2$

$p(4)=0$: $\quad a\cdot 4^2 + b\cdot 4 + c = 0$

$$\begin{cases} c = 0 \\ 4a + 2b + c = -2 \\ 16a + 4b + c = 0 \end{cases}$$

By TI84: $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & -2 \\ 16 & 4 & 1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{array}\right]$

Says that $(a, b, c) = (\frac{1}{2}, -2, 0)$ so that

$$p(x) = \frac{1}{2}x^2 - 2x = \frac{1}{2}x(x-4)$$

and one can check that $p(0) = 0 = p(4)$ and $p(2) = -2$ as required.