2.2 and 2.3 Matrix operations and Inverse matrices

ex: Solve $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$,

and $\mathbf{b} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

Claim: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc \neq 0$,

then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

In our example, $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ $ad - bc = (3)(6) - (5)(4)$

$= 18 - 20 = -2$

So $A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix}$.

NOTE: $A^{-1}A = \begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} -9+10 & -15+15 \\ 6-6 & 10-9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

ex (cont'd):

$$\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix}\begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -18 - 15/2 \\ 12 + 9/2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -51/2 \\ 33/2 \end{bmatrix} = \begin{bmatrix} -25.5 \\ 16.5 \end{bmatrix}$$

Check.

$$\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} -51/2 \\ 33/2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Defn: An $n \times n$ matrix A is invertible (or nonsingular) when there is a matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n.

Remark: Not every square matrix has an inverse.
If A has no inverse, it is not invertible (or singular).

<u>Theorem</u> 2.7: If a matrix A has an inverse, then it is unique.

<u>proof</u>: Suppose that A is invertible and and B and C are <u>each</u> inverses of A.

To <u>show</u>: $B = C$.

By definition of inverse $AB = I = BA$.

Also $AC = I = CA$

So since, $AB = I$ multiply on the ~~right~~ left by C:

$C(AB) = CI$ multiplication is associative, I = identity

$(CA)B = C$

$IB = C$ C is <u>an</u> inverse of A

$B = C$ since I = identity. $\square$

<u>Notation</u>: We denote <u>the</u> inverse of A by A^{-1}.

<u>How to find A^{-1}?</u>

<u>ex</u>: $A = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$ If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ <u>is</u> an inverse

then $AB = I$ that is $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

which has the form $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \left[c_1 \,\middle|\, c_2 \right] = \left[\begin{matrix} 1 \\ 0 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix} \right]$

By looking at how matrix multiplication works this means

1st: $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 2nd: $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solve each of these by Gaussian Elimination:

1st: $\begin{cases} 3a + 5c = 1 \\ 4a + 6c = 0 \end{cases}$ $\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 4 & 6 & 0 \end{array}\right]$

$R_2 + (-1)R_1 \to R_2$ $\begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ $\begin{bmatrix} 1 & 1 & -1 \\ 3 & 5 & 1 \end{bmatrix}$

$R_2 + (-3)R_1 \to R_2$ $\left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 2 & 4 \end{array}\right]$ $\frac{1}{2}R_2 \to R_2$ $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

$R_1 + (-1)R_2 \to R_1$ $\left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right]$ so $a = -3$, $c = 2$

Now do the same with 2nd: $\begin{cases} 3b + 5d = 0 \\ 4b + 6d = 1 \end{cases}$ $\left[\begin{array}{cc|c} 3 & 5 & 0 \\ 4 & 6 & 1 \end{array}\right]$

Observe: we could do exactly the same row operations.

To be more efficient we could solve both simultaneous systems of equations simultaneously. As follows:

ex (cont'd) : $\left[\begin{array}{cc:cc} 3 & 5 & 1 & 0 \\ 4 & 6 & 0 & 1 \end{array}\right]$

$\xrightarrow{\text{rref}} \left[\begin{array}{cc:cc} 1 & 0 & -3 & 5/2 \\ 0 & 1 & 2 & -3/2 \end{array}\right]$ so $A^{-1} = \begin{bmatrix} -3 & 5/2 \\ 2 & -3/2 \end{bmatrix}$

<u>Remark</u>: ~~By~~ To find A^{-1} (if it exists) for an $n \times n$ matrix A

we can imitate this example:

Augment A with I_n = n^{th} order identity matrix,

then do gauss-jordan elimination.

$$\left[\begin{array}{c:c} A & I_n \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{c:c} I_n & A^{-1} \end{array}\right]$$

That is, <u>if</u> the left portion of the RREF looks like I_n

then the right portion will be A^{-1}.

Find A^{-1}:

14) $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

$[A|I] = \left[\begin{array}{ccc:ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 1 & 0 \\ -1 & -4 & -7 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{array}\right]$

so $A^{-1} = \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$

ex: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ has no inverse

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + (-2)R_1 \to R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

not I No inverse

because $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represents an inconsistent system

as can be seen as follows: $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← inconsistent

so $AB = I$ for NO matrix B.