2.3 Properties of inverse matrices

Theorem 2.8: If A is an invertible $n \times n$ matrix, k is a positive integer, and c is a nonzero scalar, then:

(1) $(A^{-1})^{-1} = A$

(2) $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ times}} = (A^{-1})^k$

Notation: Define $A^{-k} = (A^{-1})^k$.

(3) $(cA)^{-1} = \frac{1}{c}A^{-1}$

(4) $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.9: If A and B are invertible $n \times n$ matrices, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

idea of proof:
$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = A(BB^{-1})A^{-1}$$
$$= AIA^{-1} = AA^{-1} = I$$

Also $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$.

Thm 2.11: If A is invertible then the system of equations

$$A\mathbb{x} = \mathbb{b}$$

has a unique solution, namely,

$$\mathbb{x} = A^{-1}\mathbb{b}$$

proof: If $A\mathbb{x} = \mathbb{b}$

then $A^{-1}A\mathbb{x} = A^{-1}\mathbb{b}$

$I\mathbb{x} = A^{-1}\mathbb{b}$

$\mathbb{x} = A^{-1}\mathbb{b}$ $\square$

ex: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}$

Solve: $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$

Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$

2.4 Elementary matrices

<u>Defn</u>: An <u>elementary matrix</u> is a matrix which results from taking I_n = the $n \times n$ identity matrix and then performing exactly <u>one</u> elementary row operation (or elementary "column operation")

<u>ex</u>: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_1$ ← elementary matrix

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{10R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix} = E_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 + 3R_1 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$

<u>NOTE</u>: Each of E_1, E_2 and E_3 are invertible matrices <u>and</u> the inverse themselves are elementary matrices.

$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/10 \end{bmatrix}$ and $\quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Could be produced by these row ops applied to I: $\quad R_2 \leftrightarrow R_3 \quad\quad \frac{1}{10}R_3 \to R_3 \quad\quad R_2 + (-3)R_1 \to R_2$

<u>Theorem 2.12</u>: Let E be an elementary matrix produced by applying <u>one</u> elementary row operation to I_m. Let A be any $m \times n$ matrix. Then $B = EA$ is the matrix produced by applying that <u>same</u> elementary row operation to A.

<u>ex</u>:

$$\begin{array}{ccccc} E & & A & & B \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} & & \begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 1 & 8 \end{bmatrix} & = & \begin{bmatrix} 2 & 5 \\ 3 & 4 \\ 9 & 28 \end{bmatrix} \end{array}$$

$R_3 + 4R_1 \to R_3$

<u>ex</u>:

$$\begin{array}{ccccc} E & & A & & B \\ \begin{bmatrix} 1 & 0 \\ 0 & 33 \end{bmatrix} & & \begin{bmatrix} 5 & 4 & 3 \\ 0 & -4 & 2 \end{bmatrix} & = & \begin{bmatrix} 5 & 4 & 3 \\ 0 & -132 & 66 \end{bmatrix} \end{array}$$

$33R_2 \to R_2$

<u>ex</u>:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 & 4 & 8 & 7 \\ 16 & 17 & 12 & 13 \\ 25 & 23 & 20 & 27 \end{bmatrix} = \begin{bmatrix} 25 & 23 & 20 & 27 \\ 16 & 17 & 12 & 13 \\ 5 & 4 & 8 & 7 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

<u>Theorem 2.14</u>: A square matrix is invertible if and only if it can be written as a product of elementary matrices.

<u>ex</u>: $A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$

<u>Given</u>: A is invertible. Here is how to write it as a product of elementary matrices.

$E_1 A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$ where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$E_2 E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ where $E_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$

$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ where $E_3 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

so
$$E_3 E_2 E_1 A = I$$
$$E_2 E_1 A = E_3^{-1}$$
$$E_1 A = E_2^{-1} E_3^{-1}$$
$$A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

That is, $A = (E_3 E_2 E_1)^{-1} I = E_1^{-1} E_2^{-1} E_3^{-1}$, and since the <u>inverses</u> of elementary matrices are elementary matrices, then we have written A as the product of elementary matrices. <u>Note</u>: The answer is <u>not</u> unique.

[Added after the end of class]

example: Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & 0 & 3 \end{bmatrix}$. Show that A can be written as the product of elementary matrices, hence, A is invertible.

$$E_1A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & -4 \\ 4 & 0 & 3 \end{bmatrix} \quad \text{where } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2E_1A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & -4 \\ 0 & -6 & 1 \end{bmatrix} \quad \text{where } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & -4 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{"} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$E_4E_3E_2E_1A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & -4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

$$E_5E_4 \cdots A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_6E_5 \cdots A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"} \quad E_6 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7E_6 \cdots A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"} \quad E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_8E_7 \cdots A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{"} \quad E_8 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_9E_8 \cdots E_1A = I \quad \text{where } E_9 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A = E_1^{-1}E_2^{-1} \cdots E_9^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Remark</u>: If A is any (not necessarily square) matrix, then there are elementary matrices $E_1, E_2 \ldots E_k$ such that,

$$E_k E_{k-1} \cdots E_3 E_2 E_1 A = R$$

where R is the reduced row-echelon form of A.

<u>ex</u>: Suppose $A = \begin{bmatrix} 0 & 0 & 3 & 6 \\ 1 & 2 & 5 & 3 \end{bmatrix}$

Then $E_1 A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 0 & 3 & 6 \end{bmatrix}$ where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$E_2 E_1 A = \begin{bmatrix} 1 & 2 & 5 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ where $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

$R = E_3 E_2 E_1 A = \begin{bmatrix} 1 & 2 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ where $E_3 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$

<u>Remark</u>: If one uses a TI-84, one can "capture" the product $E_3 E_2 E_1$ as follows:

Augment A with an identity matrix:

$$[A \mid I] = \left[\begin{array}{cccc:cc} 0 & 0 & 3 & 6 & 1 & 0 \\ 1 & 2 & 5 & 3 & 0 & 1 \end{array}\right]$$

Put this in reduced row echelon form. The right-hand portion will be the product $E_3 E_2 E_1 = E$:

$$[A \mid I] \xrightarrow{\text{rref}} \left[\begin{array}{cccc:cc} 1 & 2 & 0 & -7 & -5/3 & 1 \\ 0 & 0 & 1 & 3 & 1/3 & 0 \end{array}\right] = [R \mid E]$$

where $E = E_3 E_2 E_1 = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5/3 & 1 \\ 1/3 & 0 \end{bmatrix}.$

That is, $EA = R$, where E = product of elementary matrices, is invertible.