Math 254 Wed 27 June 2018

2.4 (cont'd) Fundamental Theorem of Invertible Matrices
(or "Equivalent Conditions" theorem)

If A is $n \times n$ matrix, then the following statements are equivalent (logically equivalent):

1. A is invertible.
2. $Ax = b$ has a unique solution for every $n \times 1$ column matrix b.
3. $Ax = 0$ has only the trivial solution, where $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.
4. A is row equivalent to I_n.
5. A can be written as the product of elementary matrices.

The "contrapositive" version:

1′. A is singular (i.e. A not invertible).

2′. $Ax = b$ has either no solution or infinitely many solutions, for any column matrix b.

3′. $Ax = 0$ has a nontrivial solution (so infinitely many).

4′. A has reduced row echelon form R where $R \neq I_n$.

5′. $A = E_1 E_2 \cdots E_k R$ where $R (\neq I_n)$ is in reduced row echelon form.