

## 3.1 Determinants of a matrix

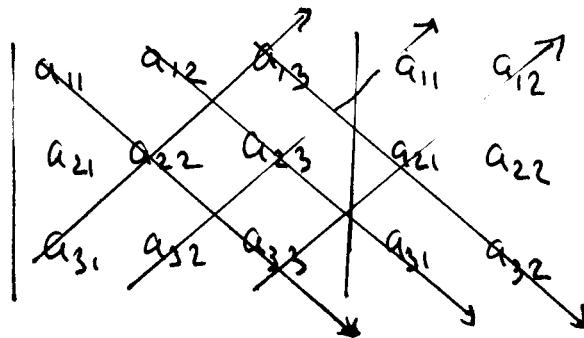
Remark: If  $A$  is an  $n \times n$  matrix, the determinant of  $A$  will be a number, denote by

$$\det(A) \text{ or } |A| .$$

Defn: If  $A = [a_{ij}]$  then  $\det(A) = a_{11}$

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then  $\det(A) = ad - bc$ .

If  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$  do this:

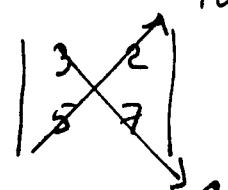


$$\begin{aligned} \text{then } \det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &\quad - a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32} \end{aligned}$$

(2)

ex if  $A = [-7]$  then  $|A| = -7$ .

If  $A = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}$  then  $\det(A) = |A| = \begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$



$$= 3 \cdot 7 - 5 \cdot 2$$

$$= 21 - 10 = 11$$

$$A = \begin{bmatrix} 3 & 5 \\ 30 & 50 \end{bmatrix} \quad \det(A) = \begin{vmatrix} 3 & 5 \\ 30 & 50 \end{vmatrix} = 3 \cdot 50 - 5 \cdot 30$$

$$= 150 - 150$$

$$= 0$$

If  $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -7 & 4 \\ -1 & -2 & -8 \end{bmatrix}$  then

$\det(A)$  is calculated as

$$\begin{vmatrix} 2 & 0 & 5 \\ 0 & -7 & 4 \\ -1 & -2 & -8 \end{vmatrix} \begin{vmatrix} 2 & 0 \\ 0 & -7 \\ -1 & -2 \end{vmatrix}$$

$$\text{so } \det(A) = 2(-7)(-8) + 0 + 0$$

$$- (-1)(-7) 5 - (-2)(4)(2) - 0$$

$$= 112 - 35 + 16 = 93$$

(3)

Defn: If  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

define the minor  $M_{ij}$  of the entry  $a_{ij}$  to be the  $2 \times 2$  determinant of the matrix produced from  $A$  by deleting the row and column containing  $a_{ij}$ .

The cofactor  $C_{ij}$  of the entry  $a_{ij}$  is

$$(-1)^{i+j} M_{ij}.$$

For example,  $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

and  $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$   
 $= -(a_{21}a_{33} - a_{23}a_{31}).$

$$\begin{array}{ccc|c} + & - & + & | \\ - & + & - & \leftarrow i+j=2+3=5 \\ + & - & + & \text{and } (-1)^5 = -1 \end{array}$$

$$i+j = 3+1 = 4$$

"checkerboard pattern"

(4)

Defn (cont'd) for the  $3 \times 3$  matrix  $A$

$$\det(A) = \sum_{j=1}^3 a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} M_{11} + (-1) a_{12} M_{12} + a_{13} M_{13}$$

This is called "expanding by cofactors in the first row."

ex: Find  $|A| = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 2 & 4 \\ 3 & 6 & 7 \end{vmatrix} = 3M_{11} - 5M_{12} + 2M_{13}$

$$= 3 \begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} - 5 \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$$

$$= 3(14 - 24) - 5(7 - 12) + 2(6 - 6)$$

$$= 3(-10) - 5(-5) + 2(0)$$

$$= -30 + 25 = -5$$

Remark: Even  $2 \times 2$  can be calculated by this method:

$$\begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} = 2|7| - 4|6| = 2(7) - 4(6)$$

$$= 14 - 24 = -10$$

(5)

Defn If  $A$  an  $n \times n$  matrix

$$\det(A) = \sum_{j=1}^n a_{1j} C_{1j}, \text{ where } C_{1j} \text{ is the cofactor of } a_{1j}$$

Note:  $C_{1j}$  is itself an  $(n-1) \times (n-1)$  determinant.

$$\text{Thm 3.1: } \det(A) = \sum_{j=1}^n a_{1j} C_{1j} \leftarrow \begin{matrix} \text{"1st row expansion"} \\ \text{OR} \end{matrix}$$

$$\text{by } \det(A) = \sum_{i=1}^n a_{ij} C_{ij} \leftarrow \text{"jth column expansion"}$$

$$\text{ex: } A = \begin{bmatrix} 0 & 5 & 3 & 1 \\ 0 & 4 & 0 & 0 \\ 7 & 8 & 4 & 2 \\ 0 & 2 & 0 & 8 \end{bmatrix}$$

$$\det(A) = 0 - 0 + 7 \begin{vmatrix} 5 & 3 & 1 \\ 4 & 0 & 0 \\ 2 & 0 & 8 \end{vmatrix} - 0 \quad \begin{matrix} \text{expand on 1st} \\ \text{column} \end{matrix}$$

$$= 7 \left( (-4) \begin{vmatrix} 3 & 1 \\ 0 & 8 \end{vmatrix} + 0 - 0 \right) \quad \begin{matrix} \text{expand on} \\ 2nd \text{ row} \end{matrix}$$

$$= 7 [(-4)(3 \cdot 8 - 0)] = 7(-4)(3)(8) = -672$$

ex (triangular matrix )

$$A = \begin{bmatrix} 2 & 7 & -50 \\ 0 & 5 & -32 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = 2 \begin{vmatrix} 5 & -32 \\ 0 & 13 \end{vmatrix} - 0 + 0 \\ = 2 \cdot 5 \cdot 13 = 130$$

Theorem 3.2 : If  $A$  is a triangular matrix of order  $n$ , then its determinant is the product of the diagonal entries.

ex : Find  $\det(A)$  if  $A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 100 & -2 & 0 & 0 & 0 \\ \frac{1}{5} & \sqrt{2} & 3 & 0 & 0 \\ \pi & \frac{1}{\sqrt{3}} & \frac{5}{7} & 4 & 0 \\ -800 & 0 & 40 & 60 & -5 \end{bmatrix}$

$$\det(A) = (5)(-2)(3)(4)(-5) \\ = 600$$

### 3.2 Determinants and elementary row operations

Theorem 3.3: Let  $A$  and  $B$  be  $n \times n$  matrices.

(1) If  $B$  is gotten from  $A$  by row swap,  
then  $\det(B) = -\det(A)$ .

(2) If  $B$  is gotten from  $A$  by multiplying  
a row by a constant  $k \neq 0$ ,  
then  $\det(B) = k \det(A)$ .

(3) If  $B$  is from  $A$  by adding a multiple of  
one row to another, then  $\det(B) = \det(A)$ .

26) Use this theorem to more easily evaluate

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{vmatrix} \xleftarrow[R_2 + -2R_1]{} \xleftarrow[R_3 - R_1]{}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & 2 \end{vmatrix} = (1)(-3)(2) = -6$$

$R_3 - R_2 \rightarrow$

Remark: The theorem is true for elementary column operations as  
well as elementary row operations.

Theorem 3.4: If  $A$  is a square matrix, then if any of the following conditions is true, then  $\det(A) = 0$ .

- (1) An entire row (or column) consists of zeros.
- (2) Two rows (or columns) are equal.
- (3) One row (or column) is a multiple of another row (or column).

ex: If  $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 4 & 8 \\ 10 & 30 & 20 \end{bmatrix}$  then  $\det(A) = 0$ .

Explanation: If  $B$  is obtained from  $A$  by  $R_3 + (-10)R_1 \rightarrow R_3$

so that  $B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$  we can

Say that  $\det(B) = 0$  by expanding by cofactors on the third row.  
 But since  $B$  is gotten from  $A$  by an elementary row operation of the "third type",  $\det(B) = \det(A)$ .