3.1 Determinants of a matrix

Remark: If A is an $n \times n$ matrix, the determinant of A will be a number, denote by

$$\det(A) \text{ or } |A| .$$

Defn: If $A = [a_{11}]$ then $\det(A) = a_{11}$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ do this:

$$\left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right.$$

then $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

ex: If $A = [-7]$ then $|A| = -7$.

If $A = \begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}$ then $\det(A) = |A| = \begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix}$ 10 21

$= 3 \cdot 7 - 5 \cdot 2$

$= 21 - 10 = 11$

$A = \begin{bmatrix} 3 & 5 \\ 30 & 50 \end{bmatrix}$ $\det(A) = \begin{vmatrix} 3 & 5 \\ 30 & 50 \end{vmatrix} = 3 \cdot 50 - 5 \cdot 30$

$= 150 - 150$

$= 0$

If $A = \begin{bmatrix} 2 & 0 & 5 \\ 0 & -7 & 4 \\ -1 & -2 & -8 \end{bmatrix}$ then

$\det(A)$ is calculated as $\left|\begin{array}{ccc|cc} 2 & 0 & 5 & 2 & 0 \\ 0 & -7 & 4 & 0 & -7 \\ -1 & -2 & -8 & -1 & -2 \end{array}\right|$

so $\det(A) = 2(-7)(-8) + 0 + 0$

$- (-1)(-7)5 - (-2)(4)(2) - 0$

$= 112 - 35 + 16 = 93$

Defn: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

define the minor M_{ij} of the entry a_{ij} to be the 2×2 determinant of the matrix produced from A by deleting the row and column containing a_{ij}.

The cofactor C_{ij} of the entry a_{ij} is

$$(-1)^{i+j} M_{ij} .$$

For example, $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

and $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$

$= -(a_{21}a_{33} - a_{23}a_{31})$.

$i+j=3$

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$i+j = 2+3 = 5$ and $(-1)^5 = -1$

$i+j = 3+1 = 4$

"checkerboard pattern"

<u>Defn</u> (cont'd) For the 3x3 matrix A

$$\det(A) = \sum_{j=1}^{3} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} M_{11} + (-1)\, a_{12} M_{12} + a_{13} M_{13}$$

This is called "expanding by cofactors in the first row."

<u>ex</u>: Find $|A| = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 2 & 4 \\ 3 & 6 & 7 \end{vmatrix} = 3M_{11} - 5M_{12} + 2M_{13}$

$$= 3 \begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} - 5 \begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$$

$$= 3(14-24) - 5(7-12) + 2(6-6)$$

$$= 3(-10) - 5(-5) + 2(0)$$

$$= -30 + 25 = -5$$

<u>Remark</u>: Even 2x2 can be calculated by this method:

$$\begin{vmatrix} 2 & 4 \\ 6 & 7 \end{vmatrix} = 2|7| - 4|6| = 2(7) - 4(6)$$

$$= 14 - 24 = -10$$

Defn If A an $n \times n$ matrix

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} \quad , \text{ where } C_{1j} \text{ is the}$$

the cofactor of a_{1j}

Note: C_{1j} is itself an $(n-1)\times(n-1)$ determinant.

Thm 3.1: $\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$ ← "ith row expansion" OR

by $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$ ← "jth column expansion"

ex: $A = \begin{bmatrix} 0 & 5 & 3 & 1 \\ 0 & 4 & 0 & 0 \\ 7 & 8 & 4 & 2 \\ 0 & 2 & 0 & 8 \end{bmatrix}$

expand on 1st column

$$\det(A) = 0 - 0 + 7 \begin{vmatrix} 5 & 3 & 1 \\ 4 & 0 & 0 \\ 2 & 0 & 8 \end{vmatrix} - 0$$

expand on 2nd row

$$= 7 \left((-4) \begin{vmatrix} 3 & 1 \\ 0 & 8 \end{vmatrix} + 0 - 0 \right)$$

$$= 7\left[(-4)\ (3 \cdot 8 - 0)\right] = 7(-4)(3)(8) = -672$$

ex (triangular matrix)

$$A = \begin{bmatrix} 2 & 7 & -501 \\ 0 & 5 & -32 \\ 0 & 0 & 13 \end{bmatrix}$$

$$\det(A) = 2 \begin{vmatrix} 5 & -32 \\ 0 & 13 \end{vmatrix} - 0 + 0$$

$$= 2 \cdot 5 \cdot 13 = 130$$

<u>Theorem 3.2</u>: If A is a triangular matrix of order n, then its determinant is the product of the diagonal entries.

ex: Find $\det(A)$ if $A = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 100 & -2 & 0 & 0 & 0 \\ \frac{1}{5} & \sqrt{2} & 3 & 0 & 0 \\ \pi & \frac{1}{\sqrt{3}} & 5/7 & 4 & 0 \\ -800 & 0 & 40 & 60 & -5 \end{bmatrix}$

$$\det(A) = (5)(-2)(3)(4)(-5)$$

$$= 600$$

3.2 Determinants and elementary row operations

Thm 3.3: Let A and B be $n \times n$ matrices.

1. If B is gotten from A by row swap, then $\det(B) = -\det(A)$.
2. If B is gotten from A by multiplying a row by a constant $k \neq 0$, then $\det(B) = k\det(A)$.
3. If B is from A by adding a multiple of one row to another, then $\det(B) = \det(A)$.

26) Use this theorem to more easily evaluate

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & -3 & -2 \end{vmatrix} \begin{matrix} \\ \leftarrow R_2 + -2R_1 \\ R_3 - R_1 \end{matrix}$$

$$= \begin{matrix} \\ \\ R_3 - R_2 \rightarrow \end{matrix} \begin{vmatrix} 1 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & 2 \end{vmatrix} = (1)(-3)(2) = -6$$

Remark: The theorem is true for elementary column operations as well as elementary row operations.

Theorem 3.4: If A is a square matrix, then if any of the following conditions is true, then $\det(A) = 0$.

1. An entire row (or column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

ex: If $A = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 4 & 8 \\ 10 & 30 & 20 \end{bmatrix}$ then $\det(A) = 0$.

Explanation: If B is obtained from A by $R_3 + (-10)R_1 \to R_3$

So that $B = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ we can

Say that $\det(B) = 0$ by expanding by cofactors on the third row.
But since B is gotten from A by an elementary row operation of the "third type", $\det(B) = \det(A)$.