Math 254 Mon 2 July 2018

3.3 Properties of determinants

ex (warmup questions) Let $A = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}$

a) write A as a product of elementary matrices.

$E_1 A = \begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ where $E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$E_2 E_1 A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ where $E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}$

$\underbrace{E_3 E_2 E_1}_{A^{-1}} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ where $E_3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

we've shown that $A^{-1} = E_3 E_2 E_1$

So $A = (A^{-1})^{-1} = (E_3 E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} E_3^{-1}$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

b) What is $\det(A)$? $\det(A) = 0 \cdot 3 - 1 \cdot 5 = -5$

(2)

c) What are the determinants of the factors (elementary matrices) from part a)?

$$|E_1^{-1}| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

$$|E_2^{-1}| = \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} = 5$$

$$|E_3^{-1}| = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1$$

d) Observe: $A = E_1^{-1} E_2^{-1} E_3^{-1}$

$$|A| = |E_1^{-1}|\,|E_2^{-1}|\,|E_3^{-1}|$$

$$\begin{vmatrix} 0 & 5 \\ 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$$5 = (-1)\quad(5)\quad(1)$$

Thm 3.5: If A and B are each $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

ex (to illustrate what will be a proof)

Again, let $A = \begin{bmatrix} 0 & 5 \\ 1 & 3 \end{bmatrix}$. Also, let $B = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$

By belief in this theorem

$$\det(AB) = |A|\,|B| = (-5)(40) = 200,$$

why should we believe this?

Recall
$$AB = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_{A} B$$

Note ①: $$\det\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B\right) = 1 \cdot \det(B)$$

this does $R_1 + 3R_2 \to R_1$, so has the same det.

Note ②: $$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B\right) = 5 \cdot \det\left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B\right)$$

this does $5R_2 \to R_2$, so multiplies the determinant by 5.

Note ③: $$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B\right) = -1 \cdot \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} B\right)$$

This does $R_1 \leftrightarrow R_2$, so toggles the sign of the determinant

$$= (-1)\cdot(5)(1)\det(B)$$
$$= (-5)\det(B) = \det(A)\det(B)$$