3.3 (cont'd) Properties of Determinants

Thm 3.6: If A is an $n \times n$ matrix, and c is a scalar, then

$$\det(cA) = c^n \det(A)$$

proof: One way we can describe scalar multiplication by c, as multiplying the "scalar matrix"

$n \times n$ matrix →

$$cI = \begin{bmatrix} c & & & 0 \\ & c & & \\ & & c & \\ 0 & & & \ddots \ c \end{bmatrix}, \text{ that is, } (cI)A = cA.$$

But now, $\det(cA) = \det((cI)A)$

$$= \det(cI) \det A$$
$$= c^n \det(A).$$

ex: $10A = 10 \begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 30 & 50 & 70 \\ 0 & 20 & 40 \\ 0 & 0 & 60 \end{bmatrix}$

Also $10A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

$\det(10A) = 10^3 \cdot 36 = 36{,}000$

Thm 3.7 A square matrix A is invertible if and only if $\det(A) \neq 0$.

proof: ($\Rightarrow$) Assume A is invertible, so $AA^{-1} = I$.

$$1 = |I| = |AA^{-1}| = |A||A^{-1}|$$

Both $|A| \neq 0$ and $|A^{-1}| \neq 0$, for otherwise $|A||A^{-1}| = 0 \neq 1$. So not only is $\det(A) \neq 0$ but also $\det(A^{-1}) \neq 0$.

($\Leftarrow$) Assume $\det(A) \neq 0$ To show: A is invertible.

Let's prove this by contradiction, that is, assume that A is not invertible. By the Fundamental Thm of Invertible matrices, we can write

$$E_k E_{k-1} \cdots E_2 E_1 A = R \quad = \text{reduced row echelon form of } A$$

where $R \neq I_n$, hence R has a row of zeros.

But now, take determinants of both sides and use the theorem on determinants of products

$$\underbrace{|E_k||E_{k-1}| \cdots |E_2||E_1|}_{\text{None of these are zero}} |A| = \underbrace{|R| = 0}_{R \text{ has a row of zeros}}$$

It must be then that $|A| = 0$, contradicting our assumption that $|A| \neq 0$.

Thm 3.8: If A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

proof: $AA^{-1} = I$

$$|A|\,|A^{-1}| = |AA^{-1}| = |I| = 1$$

these are each nonzero so $|A^{-1}| = \dfrac{1}{|A|}$.

ex #33) $A = \begin{bmatrix} k-1 & 2 \\ 2 & k+2 \end{bmatrix}$ For what values of k is A singular (i.e. NOT invertible).

Method: Find values of k such that $|A| = 0$.

$$
\begin{aligned}
0 = |A| &= (k-1)(k+2) - (2)(2) \\
&= k^2 + 2k - k - 2 - 4 \\
&= k^2 + k - 6 \\
0 &= (k+3)(k-2)
\end{aligned}
$$

when $k = -3$ or $k = 2$.

So A is NOT invertible if and only if k takes on either of these two values.

4.1 Vectors in $\mathbb{R}^n$

<u>Remark</u>: Addition and scalar multiplication can be extended in an obvious way from $\mathbb{R}^2$ or $\mathbb{R}^3$ to $\mathbb{R}^n$ = set of ordered n-tuples of real numbers.

One can prove...

<u>Theorem 4.2</u>: Properties of operations in $\mathbb{R}^n$

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^n$, and let c and d be scalars (i.e. real numbers),

properties of addition

1. $u+v$ is in $\mathbb{R}^n$ "Closure under addition"
2. $u+v = v+u$ "Commutive property"
3. $(u+v)+w = u+(v+w)$ "Associative property"
4. $\vec{u}+\vec{0} = \vec{u}$ "Additive identity"
5. $\vec{u}+(-\vec{u}) = \vec{0}$ "Additive inverse"

properties of scalar multiplication

6. $c\vec{u}$ is in $\mathbb{R}^n$ "Closure under scalar multiplication"
7. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ "Distrobutive property"
8. $(c+d)\vec{u} = c\vec{u}+d\vec{u}$ "Distributive property"
9. $c(d\vec{u}) = (cd)\vec{u}$ "Associative property of multiplication"
10. $1(\vec{u}) = \vec{u}$ "Multiplicative identity"