4.1 Vectors in R^n: linear combinations of vectors

50) Write $\vec{v}$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ where

$\vec{v} = (7, 2, 5, -3)$

$\vec{u}_1 = (2, 1, 1, 2)$

$\vec{u}_2 = (-3, 3, 4, -5)$

$\vec{u}_3 = (-6, 3, 1, 2)$.

That is, find scalars c_1, c_2, c_3, c_4 such that

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 = \vec{v}$$

Use matrix notation. Solve:

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 3 \\ 4 \\ -5 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \\ -3 \end{bmatrix}$$

Remark: This is, in effect, just a system of 4 equations, 3 variables

$$\begin{cases} 2c_1 - 3c_2 - 6c_3 = 7 \\ c_1 + 3c_2 + 3c_3 = 2 \\ c_1 + 4c_2 + c_3 = 5 \\ 2c_1 - 5c_2 + 2c_3 = -3 \end{cases}$$

ex (cont'd)
Augmented matrix :

$$\left[\begin{array}{ccc|c} 2 & -3 & -6 & 7 \\ 1 & 3 & 3 & 2 \\ 1 & 4 & 1 & 5 \\ 2 & -5 & 2 & -3 \end{array}\right] \xrightarrow{\text{rref}} \text{on TI-84}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c_1 = 2$

$c_2 = 1$

$c_3 = -1$

Check:

$$2\begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ 4 \\ -5 \end{bmatrix} - \begin{bmatrix} -6 \\ 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 5 \\ -3 \end{bmatrix} \quad \checkmark$$

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4.2 Definition of Vector Space

Let V be a set on which there are two operations, called vector addition and scalar multiplication, are defined. If the following axioms are satisfied, we say that V is a vector space.

1. $\vec{u}+\vec{v}$ is in V — "closure under addition"
2. $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ — "commutative"
3. $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ — "associative"
4. V has a zero vector $\vec{0}$ such that $\vec{u}+\vec{0}=\vec{u}$ for any $\vec{u}$ in V — "additive identity"
5. For every $\vec{u}$ in V, there is a vector $-\vec{u}$ such that $\vec{u}+(-\vec{u}) = \vec{0}$ — "additive inverse"
6. $c\vec{u}$ is in V — "closure under scalar multiplication"
7. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ — "distributive property"
8. $(c+d)\vec{u} = c\vec{u}+d\vec{u}$ — "
9. $c(d\vec{u}) = (cd)\vec{u}$ — "associative property"
10. $1(\vec{u}) = \vec{u}$ — "scalar identity"

Some examples and some non-examples of vector spaces

ex: 3×4 matrices or $M_{3,4}$. That is

$$M_{3,4} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} ; \ a_{ij} \in \mathbb{R}, \text{ for each } i,j \right\}$$

"vector addition" will be the usual addition of matrices.

"scalar multiplication" will be the usual scalar mult. of matrices.

for example:

$$5\begin{bmatrix} 2 & 0 & 4 & 8 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix} + 10\begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 & 20 & 40 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 25 & 10 \end{bmatrix} + \begin{bmatrix} 40 & 0 & 0 & 10 \\ 0 & 30 & 0 & 0 \\ 20 & 50 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 50 & 0 & 20 & 50 \\ 5 & 30 & 0 & 0 \\ 20 & 50 & 25 & 10 \end{bmatrix}$$

Check axioms

Addition

1. closure ✓
2. Commutative ✓
3. associative ✓
4. additive identity $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ✓
5. additive inverse ✓

Scalar mult

6. closure ✓
7. distributive properties ✓
8. distributive properties ✓
9. associative ✓
10. 1 = identity

ex: (Polynomials of degree 3 or less)

$$P_3 = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0, a_1, a_2, a_3 \in \mathbb{R} \}$$

vector addition: $p(x) = 3 + 2x - 5x^2 + x^3$

$q(x) = 4 - x + 2x^2 + 2x^3$

$$p(x) + q(x) = 7 + x - 3x^2 + 3x^3$$

$$10\,p(x) = 30 + 20x - 50x^2 + 10x^3$$

What is the "zero vector"? $p(x) = 0 + 0x + 0x^2 + 0x^3$

or $p(x) = 0$

"Additive inverse?" If $q(x) = 4 - x + 2x^2 + 2x^3$

then $-q(x) = -4 + x - 2x^2 - 2x^3$

$\therefore$ P_3 is a vector space.

ex (nonexample) Polynomials of degree exactly 3, with standard operations.

ex: Are the integers a vector space?

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

with scalars $= \mathbb{R}$, usual addition, scalar multiplication.

(6) fails. $\frac{1}{2}$ is a scalar,

5 is an integer

$\frac{1}{2}(5)$ is not an integer.

Here, $\mathbb{Z}$ is not closed under scalar multiplication.