4.2 More examples and nonexamples of vector spaces

example: The set $V = \mathbb{R}^2$, with standard addition but with a nonstandard scalar multiplication:

$$c(x_1, x_2) = (cx_1, 0)$$

Is V a vector space, with this weird scalar multiplication?

Note: Axioms (1)-(5) hold, as we have the usual addition.

(6) ✓ closed
(7) $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$ ✓
(8) (exercise) ✓
(9) ✓
(10) fails!

Scratch work: (7) Let $\vec{u} = (x_1, x_2)$ $\vec{v} = (y_1, y_2)$

$$\vec{u}+\vec{v} = (x_1+y_1, x_2+y_2)$$

$$c(\vec{u}+\vec{v}) = (c(x_1+y_1), 0)$$

$$c\vec{u}+c\vec{v} = (cx_1, 0) + (cy_1, 0)$$

$$= (cx_1+cy_1, 0) = (c(x_1+y_1), 0)$$

equal

(9) $d\vec{u} = d(x_1, x_2) = (dx_1, 0)$

$$c(d\vec{u}) = c(dx_1, 0) = (cdx_1, 0)$$

on the other hand: $(cd)\vec{u} = (cd)(x_1, x_2) = (cdx_1, 0)$

equal

Example 8 (continued) Axiom (10) fails: Axiom (10) says: $1(\vec{u}) = \vec{u}$, for any $\vec{u}$.

A counterexample: Suppose $\vec{u} = (0, 5)$

$$1\vec{u} = \left((1)(0), 0\right) = (0,0) \neq (0,5)$$

That $1\vec{u} \neq \vec{u}$

$\therefore$ This set V with exotic scalar multiplication is not a vector space.

4.2 #31) Let V = 2×2 singular = (non-invertible) matrices, with the usual addition and scalar multiplication.

Axiom (5): Is there a zero vector in V?

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the zero vector, as it "acts like" the zero vector and it is in V since $\det \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

Axiom (1): Are singular matrices closed under addition?

No: Counterexample: If $A = \begin{bmatrix} 5 & 10 \\ 2 & 4 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then

$|A| = 0$ and $|B| = 0$ But $|A+B| = \begin{vmatrix} 6 & 10 \\ 2 & 4 \end{vmatrix} = 4 \neq 0$

So A and B are in V, but A+B is not in V.

The set V is not closed under addition.

4.3 Subspaces of Vector spaces

Defn: A nonempty subset W of a Vector space V is a subspace of V when W is a vector space using the operation of addition and Scalar multiplication it inherits from V.

Thm 4.5 "Test for a subspace"

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the two closure axioms hold:

(1) if $\vec{u}$ and $\vec{v}$ are in W then $\vec{u}+\vec{v}$ is in W.

(6) If $\vec{u}$ is in W, then $c\vec{u}$ is in W.

Remark (1): In real life, the vast majority of vector space we will deal with will be subspaces of known vector spaces.

(2) To prove that subset W of a vector space V is NOT a subspace of V, it may be easier to show that one of the axioms besides the closure axioms fails.

ex: Let W = 3×3 symmetric matrices.

Recall: A matrix A is <u>symmetric</u> if $A = A^T$.

<u>Note</u>: For $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ $A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

so if $A = A^T$, then $b = d$, $c = g$, $f = h$.

ex: $A = \begin{bmatrix} 3 & 5 & 2 \\ 5 & 7 & 4 \\ 2 & 4 & 11 \end{bmatrix}$ is symmetric

Is W a subspace of $M_{3,3}$? Answer: <u>Yes</u>.

<u>Facts</u>: (1) $(A+B)^T = A^T + B^T$

(2) $(cA)^T = c(A^T)$

<u>proof</u>: Suppose A and B are in W. That is:

$A^T = A$ and $B^T = B$.

Then $(A+B)^T = A^T + B^T = A + B$,

so $A+B$ is also symmetric.

Suppose $A = A^T$.

Then $(cA)^T = c(A^T) = cA$

so cA is also symmetric.

$\therefore$ W <u>is</u> a subspace of $M_{3,3}$.

[Added after class ended.]

4.3 <u>More examples of subspaces</u> (or non-subspaces)

4) If W is the subset of $M_{3\cdot 2}$ of the form $\begin{bmatrix} a & b \\ a-2b & 0 \\ 0 & c \end{bmatrix}$

verify that W is a subspace of $M_{3,2}$.

<u>closed under addition</u>: Let A and B be two elements of W, say,

$A = \begin{bmatrix} a_1 & b_1 \\ a_1-2b_1 & 0 \\ 0 & c_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ a_2-2b_2 & 0 \\ 0 & c_2 \end{bmatrix}$. To show: $A+B$ is in W:

$$A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ (a_1+a_2)-2(b_1+b_2) & 0 \\ 0 & c_1+c_2 \end{bmatrix} \text{ has the form } \begin{bmatrix} a_3 & b_3 \\ a_3-2b_3 & 0 \\ 0 & c_3 \end{bmatrix}$$

where $a_3 = a_1+a_2$, $b_3 = b_1+b_2$, and $c_3 = c_1+c_2$, hence is in W.

<u>closed under scalar multiplication</u>: Let A be an element of W and

Let k be a scalar. Then

$$kA = k\begin{bmatrix} a_1 & b_1 \\ a_1-2b_1 & 0 \\ 0 & c_1 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 \\ ka_1-2kb_1 & 0 \\ 0 & kc_1 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ a_2-2b_2 & 0 \\ 0 & c_2 \end{bmatrix} \text{ is in } W$$

where $a_2 = ka_1$, $b_2 = kb_1$, $c_2 = kc_1$

8) Show that W = vectors in $\mathbb{R}^2$ whose first component is 2 is <u>NOT</u> a subspace of $\mathbb{R}^2$. That is, $W = \{(2,y) : y \in \mathbb{R}\}$.

Several possible answers: W lacks an additive identity (ie Axiom 4 fails)

or, If $(2,y_1)$ and $(2,y_2)$ are in W, then

$(2,y_1)+(2,y_2) = (4, y_1+y_2)$ is not in W (Axiom 1 fails)

or If $(2,y)$ is in W, then $10(2,y) = (20, 20y)$ is not in W (i.e. Axiom 6 fails).

32) Let $W = \{ A \in M_{n,n} : AB = BA \}$ where **B** is a fixed $n\times n$ matrix.
Is W a subspace of $M_{n,n}$? Answer: Yes.

Closed under addition: Let A_1 and A_2 be elements of W, so that

$$A_1 B = BA_1 \quad \text{and} \quad A_2 B = BA_2 .$$

Then $(A_1+A_2)B = A_1 B + A_2 B = BA_1 + BA_2 = B(A_1+A_2)$
so that $A_1 + A_2$ is in W.

Closed under scalar multiplication: Let A be in W. Then $AB = BA$.
If c is a scalar, $(cA)B = c(AB) = c(BA) = B(cA)$
hence cA is also in W.

34) Let W = set of $n\times n$ invertible matrices. Is W a subspace of $M_{n,n}$?
Answer: No. Reason: W has no additive identity (Axiom 4 fails).
Alternatively, W is not closed under addition: in $M_{2,2}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ are invertible,}$$

but $A + B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

48) The set $S = \{ f \in C[0,1] : \int_0^1 f(x)\,dx = 0 \}$ is a subspace of $C[0,1]$.
Prove this.
proof: closed under addition: Suppose f and g are in S.

$$\text{Then } \int_0^1 (f+g)(x)\,dx = \int_0^1 [f(x) + g(x)]\,dx = \int_0^1 f(x)dx + \int_0^1 g(x)dx = 0 + 0 = 0$$

$\therefore$ $f+g$ is in S.

Closed under scalar multiplication: Suppose f is in S and c is a scalar.

$$\int_0^1 (cf)(x)\,dx = \int_0^1 c f(x)\,dx = c \int_0^1 f(x)\,dx = c(0) = 0 .$$

$\therefore$ cf is in S.