math 254 Wed 11 Jul 2018

4.4 Spanning sets and Linear Independence

ex: In $\mathbb{R}^3$, let $\vec{u}_1 = (2,1,0)$ and let $\vec{u}_2 = (0,-1,1)$. What is the smallest subset W of $\mathbb{R}^3$ which contains $\vec{u}_1$ and $\vec{u}_2$ and makes W into a <u>subspace</u> of $\mathbb{R}^3$?

Since $\vec{u}_1 = (2,1,0)$ lies in W, so must $c_1 \vec{u} = c_1(2,1,0) = (2c_1, c_1, 0)$ where c_1 is <u>any</u> scalar.

Likewise $c_2 \vec{u}_2 = c_2(0,-1,1) = (0,-c_2,c_2)$ where $c_2 \in \mathbb{R}$, must also be in W.

Moreover, any <u>sum</u> of these vectors must be in W. Answer:

$$W = \{ c_1\vec{u}_1 + c_2\vec{u}_2 : c_1, c_2 \in \mathbb{R} \}$$

= smallest set containing $\vec{u}_1$ and $\vec{u}_2$ which is a <u>subspace</u> of $\mathbb{R}^3$.

Defn: [of a Spanning Set of a Vector Space]

Let $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ be subset of a vector space V. The set S is a spanning set of V when every vector in V can be written as a linear combination of vectors in S. We say that S spans V.

ex: In the previous example, $S = \{(2,1,0), (0,-1,1)\}$
$= \{\vec{u}_1, \vec{u}_2\}$

W was spanned by S.

Notation: If $S = \{\vec{v}_1, \vec{v}_2 \dots \vec{v}_k\}$ are vectors in V, the span of S is

$$\text{span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k\}$$

Theorem 4.7 span(S) is a subspace of V

If $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$ are vectors in a vectorspace V, then span(S) is a subspace of V. Moreover, span(S) is the smallest subspace of V that contains S. (i.e. any subspace W which contains S, must contain span(S)).

idea of proof: Show that span(S) is closed under addition and scalar multiplication.

<u>Remark</u>: What are the possible <u>subspaces</u> of $\mathbb{R}^3$?

$$\{\vec{0}\} = \{(0,0,0)\} \text{ is a subspace of } \mathbb{R}^3.$$

<u>Next.</u> Let $\vec{u}_1$ be a fixed nonzero vector.

$$\text{span}(\{\vec{u}_1\}) = \{c_1\vec{u}_1 : c_1 \in \mathbb{R}\} = \text{a line through the origin.}$$

<u>Next</u>: Suppose $\vec{u}_1$ and $\vec{u}_2$ are nonzero vectors, <u>and</u> neither is a scalar multiple of the other. (That is, $\vec{u}_1$ and $\vec{u}_2$ are "linear independent".)

$$\text{span}(\{\vec{u}_1, \vec{u}_2\}) = \{c_1\vec{u}_2 + c_2\vec{u}_2 : c_1, c_2 \in \mathbb{R}\}$$
$$= \text{a } \underline{\text{plane}} \text{ through the origin.}$$

<u>Finally</u>: $\mathbb{R}^3$ is a subspace of itself.

<u>Defn</u>: A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in a vector space V, is <u>linearly</u> <u>independent</u> when the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has <u>only</u> the trivial solution:

$$c_1 = 0, \quad c_2 = 0, \quad \ldots, \quad c_k = 0.$$

If there <u>are</u> <u>nontrivial</u> solutions of the equation, we say that S is <u>linearly</u> <u>dependent</u>.

ex: In $\mathbb{R}^3$ let $S = \{\vec{u}_1, \vec{u}_2\}$ where

$$\vec{u}_1 = (2,1,0) \quad \text{and} \quad \vec{u}_2 = (0,-1,1).$$

Is S linearly independent?

That is, is there only the trivial solution to

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 = \vec{0} \quad ?$$

That is,

$$c_1 (2, 1, 0) + c_2 (0, -1, 1) = (0, 0, 0)$$

This amounts to solving a system of 3 eqns, 2 variables:

$$\begin{cases} 2c_1 + 0c_2 = 0 \\ 1c_1 + (-1)c_2 = 0 \\ 0c_1 + 1c_2 = 0 \end{cases}$$

coefficient matrix:

$$\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

That is, $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ 0 = 0 \end{cases}$ So there is only the trivial solution.

So S is linearly independent.

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ex: In $P_2 = \{ a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in R \}$

Let
$$p_1(x) = 1 + 2x + x^2$$
$$p_2(x) = 4 \quad - x^2$$
$$p_3(x) = 10 + 4x$$

Is the set $S = \{ p_1(x), p_2(x), p_3(x) \}$ linearly independent?

zero polynomial

$$c_1 p(x) + c_2 p_2(x) + c_3 p_3(x) = 0$$

$$c_1(1 + 2x + x^2) + c_2(4 \quad - x^2) + c_3(10 + 4x) = 0$$

$$(c_1 + 4c_2 + 10c_3) + (2c_1 + 4c_3)x + (c_1 - c_2)x^2 = 0$$

$$\begin{cases} c_1 + 4c_2 + 10c_3 = 0 \\ 2c_1 \qquad + 4c_3 = 0 \\ c_1 - c_2 \qquad = 0 \end{cases}$$

$$\begin{bmatrix} 1 & 4 & 10 \\ 2 & 0 & 4 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} c_1 + 2c_3 = 0 \\ c_2 + 2c_3 = 0 \end{cases}$$

Let $t = c_3$

then $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2t \\ -2t \\ t \end{bmatrix}$

in particular (if $t = 1$) $(c_1, c_2, c_3) = (-2, -2, 1)$ is a nontrivial solution so S is linearly dependendent.

[Added after class]

<u>Remark</u>. In the previous example, we saw that $S = \{p_1(x), p_2(x), p_3(x)\}$ is a <u>linearly</u> <u>dependent</u> set because

$$-2p_1(x) - 2p_2(x) + p_3(x) = 0$$

that is,

$$-2(1+2x+x^2) - 2(4-x^2) + (10+4x) = 0$$

or, in words, there a nontrivial linear combination of the polynomials in S which equals the zero polynomial.

Note that this says $p_3(x) = 2p_1(x) + 2p_2(x)$, that is, $p_3(x)$ is a linear combination of the remaining vectors in S. This is an example of...

<u>Theorem 4.8</u>: A set $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\vec{v}_i$ can be written as a linear combination of the other vectors in S.

<u>proof</u>: [Read it in the textbook, and <u>understand</u> the proof.]

<u>Corollary</u>: Two vectors $\vec{u}$ and $\vec{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

<u>proof</u> [directly, without use of Thm 4.8]: ($\Rightarrow$) Suppose $S = \{\vec{u}, \vec{v}\}$ is linearly dependent. Then there are scalars c_1, c_2 not both 0, such that $c_1\vec{u} + c_2\vec{v} = \vec{0}$. If $c_1 \neq 0$, then $\vec{u} + \frac{c_2}{c_1}\vec{v} = 0 \Rightarrow \vec{u} = -\frac{c_2}{c_1}\vec{v}$.

If $c_2 \neq 0$, then $\frac{c_1}{c_2}\vec{u} + \vec{v} = \vec{0} \Rightarrow \vec{v} = -\frac{c_1}{c_2}\vec{u}$.

($\Leftarrow$) If $\vec{u} = c\vec{v}$ for some scalar $c \neq 0$, then $-\vec{u} + c\vec{v} = \vec{0}$.

If $\vec{v} = c\vec{u}$ for some scalar $c \neq 0$, then $c\vec{u} - \vec{v} = \vec{0}$.

In either case, this shows that S is linearly dependent. Finally, if $\vec{v} = c\vec{u}$ or $\vec{u} = c\vec{v}$ where $c = 0$, this says $\vec{v}$ (or $\vec{u}$) $= \vec{0}$, and any set containing the zero vector must be linearly dependent. (Why? Exercise.)