4.5 Basis and Dimension

Defn: A set of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ in a vector space V is a basis for V when these two conditions are true:

1. S spans V
2. S is linearly independent.

Examples of "standard bases"

ex: In $\mathbb{R}^3$ let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard basis for $\mathbb{R}^3$.

(1) Is $\text{span}(S) = \mathbb{R}^3$. Let $\vec{x} = (a, b, c)$ be an arbitrary vector in $\mathbb{R}^3$.

Can we write $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

for some scalars c_1, c_2, c_3? Well, sure.

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

ex (cont'd)

(2) Is S linearly independent?

For

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is there a nontrivial solution? Nope:

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ is the solution.

ex: P_3 = cubic (~~or~~ less) polynomials $= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \}$

$S = \{ 1, x, x^2, x^3 \}$ = standard basis

ex: $M_{2,2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

4.5 8) In $\mathbb{R}^2$, $S = \{ (2,3), (6,9) \}$.

Why is S not a basis for $\mathbb{R}^2$?

Answer: S is linearly dependent:

$$3(2,3) - 1(6,9) = (0,0)$$

Remark: It is also true that $\operatorname{span}(S) \neq \mathbb{R}^2$ but it's less obvious.

26) In P_2 let $S = \{-1,\ 11x\}$.

Why is S not a basis for P_2?

Despite that S is linearly independent, S is NOT a basis for P_2 because $\text{span}(S) \neq P_2$. To see this, note that $x^2 \in \text{span}(S)$. Why?

$$c_1(-1) + c_2(11x)$$

will be linear, constant or zero, hence NOT x^2.

48) In P_3, for $S = \{4t - t^2,\ 5 + t^3,\ 5 + 3t,\ -3t^2 + 2t^3\}$

is S a (nonstandard) basis for P_3?

[Let's hold off on this until ~~some~~ we see some handy theorems.]

Theorem 4.9 : "Uniqueness of a basis representation"

If $S = \{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Remark: "One" because S spans V.

"Only one" because is S linearly independent.