Math 254

§4.5 Basis and dimension (cont'd)

(Special case $n=3$)

Thm 4.9: If $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for a vector space V, then every vector in $\vec{u}$ in V can be written in one and only one way as

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 .$$

Remark: That is, once we have settled on a basis S for a vector space V, and given $\vec{u}$ in V, there is one and only one set of scalars c_1, c_2, c_3 such that $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$.

We will soon call c_1, c_2, c_3 the "coordinates of the $\vec{u}$ with respect the basis S for V."

proof (for $n=3$) (part 1): Since S spans V and $\vec{u} \in V$, we know that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

because S is a spanning set for V, so there is one way...

(part 2): Why is there only one way to write $\vec{u}$ as a linear combination of vectors in S? Suppose that

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \quad \text{for scalars } c_1, c_2, c_3$$

but also $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3$ for b_1, b_2, b_3.

This would say

$$\vec{0} = \vec{u} - \vec{u} = (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) - (b_1\vec{v}_1 + b_2\vec{v}_2 + b_3\vec{v}_3)$$

$$= (c_1 - b_1)\vec{v}_1 + (c_2 - b_2)\vec{v}_2 + (c_3 - b_3)\vec{v}_3$$

Because S is linearly independent, the only solution is the trivial solution, so $c_1 - b_1 = 0$, $c_2 - b_2 = 0$, $c_3 - b_3 = 0$ so $c_1 = b_1$, $c_2 = b_2$, and $c_3 = b_3$. ∎

ex: In P_2 = deg. 2 or less polynomials
with $S = \{1, x, x^2\}$ = std. basis

Suppose $\vec{u} = 5 + 7x - 3x^2$. Here

if $c_1 = 5$, $c_2 = 7$, $c_3 = -3$

then $\vec{u} = c_1 1 + c_2 x + c_3 x^2$ is unique
representation of $\vec{u}$ as a linear combination of $1, x, x^2$.

Theorem 4.10: If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors must be linearly dependent.

Heart of the proof: when you have a linear system, homogenous, with more variables than equations, there must be a nontrivial solution. Reason: In the r.r.e.f. of a matrix with more columns than rows, there must be a column without a leading 1.

ex: #24) P_2 = quadratic or less polynomials, has a basis with three elements, $\{1, x, x^2\}$ = standard basis

If $S = \{2, x, 3+x, 3x^2\}$

by this theorem, simply because $4 > 3$,

we can say that S must be linearly dependent.

Remark: But we will not use this theorem, but will instead show "the hard way" that S is linearly dependent. (This will give insight into how the proof works.)

#24 (cont'd) "the hard way," using notation like that of the proof of Theorem 4.10.

Let $\vec{v}_1 = 1$, $\vec{v}_2 = x$, $\vec{v}_3 = x^2$

So that $\{v_1, v_2, v_3\}$ = a basis for P_2.

Let $\vec{u}_1 = 2$, $\vec{u}_2 = x$, $\vec{u}_3 = 3+x$, $\vec{u}_4 = 3x^2$.

That is (by Thm 4.9)

$$\vec{u}_1 = 2\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = (2)1 + 0x + 0x^2$$
$$\vec{u}_2 = 0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 = (0)1 + 1x + 0x^2$$
$$\vec{u}_3 = 3\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 = (3)1 + 1x + 0x^2$$
$$\vec{u}_4 = 0\vec{v}_1 + 0\vec{v}_2 + 3\vec{v}_3 = (0)1 + 0x + 3x^2$$

Is S linear independent, or not? Are there scalars k_1, k_2, k_3, k_4

Such that: $$k_1\vec{u}_1 + k_2\vec{u}_2 + k_3\vec{u}_3 + k_4\vec{u}_4 = \vec{0} \quad ?$$

That is:
$$\begin{aligned} & k_1(2\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3) \\ + & k_2(0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3) \\ + & k_3(3\vec{v}_1 + 1v_2 + 0\vec{v}_3) \\ + & k_4(0\vec{v}_1 + 0\vec{v}_2 + 3\vec{v}_3) = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \end{aligned}$$

By the <u>uniqueness</u> of the representations as a linear combination (Thm 4.9),

This says:
$$2k_1 + 0k_2 + 3k_3 + 0k_4 = 0 \quad (\vec{v}_1 \text{ coefficients})$$
$$0k_1 + 1k_2 + 1k_3 + 0k_4 = 0 \quad (\vec{v}_2 \text{ "})$$
$$0k_1 + 0k_2 + 0k_3 + 3k_4 = 0 \quad (\vec{v}_3 \text{ "})$$

coefficient matrix: $$\begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ so that $$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t \\ -t \\ t \\ 0 \end{bmatrix}$$

... is the solution set. If $t=2$, then $\vec{0} = -3\vec{u}_1 - 2\vec{u}_2 + 2\vec{u}_3 + 0\vec{u}_4$
$= -3(2) - 2(x) + 2(3+x) = 0$

↓ Added after class.

Some similar problems but done, instead, "the easy way," i.e. using Theorem 4.10.

§4.5 #12) Why is $S = \{(-1,2), (1,-2), (2,4)\}$ NOT a basis for $\mathbb{R}^2$.

Answer: Because $\mathbb{R}^2$ has a basis (the standard basis) with 2 elements, any set of vectors, like S, having 3 or more vectors must be linearly dependent, hence not a basis for $\mathbb{R}^2$.

#22) Why is $S = \{(6,4,1), (3,-5,1), (8,13,6), (0,6,9)\}$ not a basis for $\mathbb{R}^3$? Because $\mathbb{R}^3$ has a basis with 3 elements in it, and S has 4, and $4 > 3$.

#18) Why is $S = \{(1,1,2), (0,2,1)\}$ not a basis for $\mathbb{R}^3$?

Suppose S were a basis for $\mathbb{R}^3$. By Theorem 4.10, this would imply that the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ would be linearly dependent (since $3 > 2$), contradicting that the standard basis for $\mathbb{R}^3$ is linearly independent. Hence, S cannot be a basis for $\mathbb{R}^3$.

Theorem 4.11 "The Number of Vectors in a Basis"

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

proof: Let $S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and let $S_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ be any other basis for V.
By Theorem 4.10, if $m > n$ = number in basis S_1, that would imply that S_2 is linearly dependent, contradicting that S_2 is a basis for V, so $m \le n$.
Likewise by Thm 4.10, if $n > m$ = number in basis S_2, then S_1 would be linearly dependent, contradicting that S_1 is a basis for V, so $n \le m$.
$\therefore$ $n = m$.