Math 254 Wed 18 July 2018

4.6 Row space, column space, and nullspace of a matrix

<u>Defn</u>: Let A be an $m \times n$ matrix

1) The <u>row space</u> of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
2) The <u>column space</u> of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A.

<u>ex</u>: $A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & 4 \end{bmatrix}$

row space of A = span $\{(0,1,-1), (-2,3,4)\}$
= a subspace of $\mathbb{R}^3$

column space of A = span $\{(0,-2), (1,3), (-1,4)\}$
= a subspace of $\mathbb{R}^2$

Thm 4.13 "Row equivalent matrices have the same row space"

If an $n \times m$ matrix is row-equivalent to an $n \times m$ matrix B, then the row space of A and of B are equal.

<u>Idea of proof</u>: Because rows of B are linear combinations of rows of A, every row of B is in row space of A. Now, every linear combination of rows of B will then be a linear combination of rows of A, so row space$(B) \subseteq$ rowspace(A). Likewise, rowspace$(A) \subseteq$ rowspace(B). $\therefore$ rowspace(A) = rowspace(B).

Thm 4.14 "Basis for the Row Space of a Matrix"

If a matrix A is row equivalent to matrix B, where B is in reduced row-echelon form, then the nonzero row vectors of B form a basis for the rowspace of A.

ex: #10) $A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix} \xrightarrow[\text{by TI-84}]{\text{rref}}$

$B = \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}$

Let $\vec{w}_1 = (1, 0, 4/5)$, $\vec{w}_2 = (0, 1, 1/5)$

then if $S = \{\vec{w}_1, \vec{w}_2\}$ then S is a basis for the row space of A. That is:

1) These two nonzero rows of B are linearly independent.

and 2) span (rows of A) = span (nonzero rows of B), by theorem 4.13, since A and B are row equivalent.

Defn: The row rank of a matrix A = dim (row space of A)
= number of leading 1s of the red. rowechelon form of A.

In this example: row rank = 2.

<u>ex</u> : (How to find a basis for the <u>column</u> space of A)

<u>Method 1</u> : Take the transpose of A, then find a basis for the row space of A^T.

Use A from the previous example :

$$A^T = \begin{bmatrix} 2 & 5 & 8 \\ -3 & 10 & -7 \\ 1 & 6 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 23/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{bmatrix}$$

a basis for column space of A

$$= \left\{ \left(1, 0, \tfrac{23}{7}\right), \left(0, 1, \tfrac{2}{7}\right) \right\}$$

<u>Method 2</u>: A is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Ask : which columns have a leading 1 ?

Answer: 1^{st} and 2^{nd}.

Take the <u>corresponding</u> columns of A :

$$S = \{ (2, 5, 8), (-3, 10, -7) \}$$

then S will be a basis for the column space of A.

[Why?, we'll see.]

↓ Added after class.

<u>Why "Method 2" works</u> to find a basis for the column space of A.

The main idea: If A and B are matrices which are row equivalent, then linear dependency relationships among rows of A are the <u>same</u> as those of B.

For instance, since (in the example)

$$B = \begin{bmatrix} 1 & 0 & 4/5 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{bmatrix} \qquad 5\begin{pmatrix}\text{3rd} \\ \text{column} \\ \text{of } B\end{pmatrix} = \begin{bmatrix}4\\1\\0\end{bmatrix} = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$= 4\begin{pmatrix}\text{1st} \\ \text{column} \\ \text{of } B\end{pmatrix} + 1\begin{pmatrix}\text{2nd} \\ \text{column} \\ \text{of } B\end{pmatrix}$$

the <u>same</u> linear relationship will hold for the <u>corresponding</u> rows of A:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix} \qquad 5\begin{pmatrix}\text{3rd} \\ \text{column} \\ \text{of } A\end{pmatrix} = \begin{bmatrix}5\\30\\25\end{bmatrix} = 4\begin{bmatrix}2\\5\\8\end{bmatrix} + \begin{bmatrix}-3\\10\\-7\end{bmatrix}$$

$$= 4\begin{pmatrix}\text{1st} \\ \text{column} \\ \text{of } A\end{pmatrix} + 1\begin{pmatrix}\text{2nd} \\ \text{column} \\ \text{of } A\end{pmatrix}$$

<u>This is not a coincidence</u>, of course. Here is why.

Denote the columns of B as $\vec{B}_1, \vec{B}_2$, and $\vec{B}_3$. The relationship among the columns of B can be written as $5\vec{B}_1 + 5\vec{B}_2 - \vec{B}_3 = \vec{0}$, or in matrix form,

$$\left[\vec{B}_1 \middle| \vec{B}_2 \middle| \vec{B}_3\right]\begin{bmatrix}5\\5\\-1\end{bmatrix} = B\begin{bmatrix}5\\5\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} = \vec{0}$$

If we apply an elementary row operation to B, that can be effected by multiplying on the left by an elementary matrix E_1:

so that $$E_1 B\begin{bmatrix}5\\5\\-1\end{bmatrix} = E_1\begin{bmatrix}0\\0\\0\end{bmatrix} = E_1\vec{0} = \vec{0}$$

~~After~~ a second elementary row operation, we would have

$$(E_2 E_1) B \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = (E_2 E_1) \vec{0} = \vec{0}$$

Then $(E_3 E_2 E_1) B \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = (E_3 E_2 E_1) \vec{0} = \vec{0}$

After k elementary row operations, so that $(E_k \cdots E_2 E_1) B = A$, we get that

$$A \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = (E_k \cdots E_2 E_1) B \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = (E_k \cdots E_2 E_1) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

That is, if the columns of A are $\vec{A}_1$, $\vec{A}_2$, and $\vec{A}_3$ the statement that $A \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = [\vec{A}_1 | \vec{A}_2 | \vec{A}_3] \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

is just that statement that $5\vec{A}_1 + 5\vec{A}_2 - \vec{A}_3 = \vec{0}$, which is the <u>same</u> linear relationship $5\vec{B}_1 + 5\vec{B}_2 - \vec{B}_2 = \vec{0}$, that holds for the columns of B.