Math 254 Mon 23 July 2018

4.6 Rank of a matrix (cont'd)

ex: Find bases for the row space and the

#8) column space of

$$A = \begin{bmatrix} 2 & 5 \\ -2 & -5 \\ -6 & -15 \end{bmatrix} \xrightarrow{\text{rref}}$$

$$B = \begin{bmatrix} 1 & 2.5 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Basis for the row space of A $= \{(1, 2.5)\} = \{(1, \tfrac{5}{2})\}$

So the row rank of A is 1.

Basis for the column space of A (method 2)

$= \{(2, -2, -6)\}$

Defn: The column rank of a matrix A = dim (column space of A).
= number of leading 1s in the reduced row echelon form of A.

Theorem 4.15: The row rank of A = the column rank of A.
That is, the dimension of the row space of A
= the dimension of the column space of A.

proof: Both numbers equal the number of leading 1s in the reduced row echelon form of A.

<u>Defn</u>: The <u>rank</u> of a matrix A is the dimension of the row space (or column space) of A.

<u>ex</u> 24) Find a basis for the column space of A, where $A = \begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix} \xrightarrow{\text{rref}}$

$$B = \begin{bmatrix} 1 & 0 & \frac{1}{4} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

[method 2] Basis for the column space of A = $\{(4,6,2), (20,-5,-11)\}$

[Another correct answer by method 1]

$$A^T = \begin{bmatrix} 4 & 6 & 2 \\ 20 & -5 & -11 \\ 31 & -6 & -16 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for column space of A = $\{(1, 0, -\frac{2}{5}), (0, 1, 3/5)\}$

<u>Remark</u>: In effect the matrix

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{2}{5} & \frac{3}{5} & 0 \end{bmatrix}$$

is <u>column</u> equivalent to A so column space of C = column space of A. Take the nonzero <u>columns</u> of C as the basis for the column space of A.
This is what method 1 is.

<u>Defn</u>: If A is an $m\times n$ matrix, the set of all solutions of the homogeneous system of equations

m equations →
n variables

$$\overset{m\times n}{A}\ \overset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \overset{m\times 1}{\left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}} \text{ m times}$$

is called the <u>nullspace</u> (or <u>solution space</u>) of A.

More compact notation: $A\vec{x} = \vec{0}$.

We denote the nullspace of A as $N(A)$.

<u>Thm</u> 4.16: The nullspace of A is a <u>subspace</u> of $\mathbb{R}^n$. We call $\dim(N(A))$ = <u>nullity of A</u>.

<u>proof</u>: (easy). The nullspace of A is <u>nonempty</u> since $\vec{0}\in N(A)$:

$$A\vec{0} = \vec{0}$$

<u>N(A) is closed under addition</u>: Suppose that $\vec{x}_1\in N(A)$ and $\vec{x}_2 \in N(A)$.

That is, $A\vec{x}_1 = \vec{0}$ and $A\vec{x}_2 = \vec{0}$. Then

$$A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}+\vec{0} = \vec{0}$$

so that $x_1 + x_2 \in N(A)$.

<u>N(A) is closed under scalar multiplication</u>: Let $\vec{x}\in N(A)$ and let c be a scalar. Then $A(c\vec{x}) = cA\vec{x} = c\vec{0} = \vec{0}$

So that $c\vec{x}\in N(A)$.

Since the nullspace of A is a nonempty <u>subset</u> of $\mathbb{R}^n$ which is closed under addition and scalar multiplication, it is a <u>subspace</u> of $\mathbb{R}^n$.