Math 254 Tue 24 July 2018

4.6 (cont'd) Null space of A

ex 38) $A = \begin{bmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{bmatrix}$ a) Find $N(A)$, and find a basis for $N(A)$.

$\xrightarrow{\text{rref}} B = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Now consider the homogeneous system

$$\underset{3\times 4}{A} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$ is equivalent to

$$\begin{cases} x_1 + x_3 - 2x_4 = 0 \\ x_2 - x_3 + 2x_4 = 0 \end{cases}$$

Let $x_3 = s$, $x_4 = t$.

$N(A)$ consists of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s + 2t \\ s - 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $N(A) = \{ (-1, 1, 1, 0), (2, 2, 0, 1) \}$

ex (cnt'd) For A an $m \times n$ matrix, so A has m rows
n columns

NOTATION: $N(A) = \text{null}(A)$ = nullspace of A
a subspace of $\mathbb{R}^n$

$\dim(\text{null}(A)) = \text{nullity}(A)$

$\text{row}(A)$ = rowspace of A
also a subspace of $\mathbb{R}^n$

$\text{col}(A)$ = column space of A
a subspace of $\mathbb{R}^m$

$\dim(\text{row}(A)) = \dim(\text{col}(A)) = \text{rank}(A)$

b) Find a basis for $\text{row}(A)$:

$$\{(1,0,1,-2),\ (0,1,-1,2)\}$$

c) Find a basis for $\text{col}(A)$:

$$\{(1,0,-2),\ (3,1,-6)\}$$

Thm 4.17 "Dimension of the solution space"
If A is an $m \times n$ matrix, the dimension of the solution space of $A\vec{x} = \vec{0}$ is $n-r$, where $r = \text{rank}(A)$. That is

$$\boxed{n = \text{rank}(A) + \text{nullity}(A)}$$

idea of proof: n = number of columns of A
$\text{rank}(A)$ = number of leading 1s in rref of A
$\text{nullity}(A)$ = number of columns of the rref without leading 1s.

Q: What about solution sets of nonhomogeneous systems of equations? $A\vec{x} = \vec{b}$

Thm 4.18

If $\vec{x}_p$ is one particular solution of $A\vec{x} = \vec{b}$, every solution of this system can be written in the form

$$\vec{x} = \vec{x}_p + \vec{x}_h$$

where $\vec{x}_h$ is the solution of the corresponding homogeneous system $A\vec{x} = \vec{0}$.

proof: Let $\vec{x}$ be any solution of $A\vec{x} = \vec{b}$.

Since $\vec{x}_p$ is one solution, that is, $A\vec{x}_p = \vec{b}$

then $A(\vec{x} - \vec{x}_p) = A\vec{x} - A\vec{x}_p = \vec{b} - \vec{b} = \vec{0}$.

Letting $\vec{x}_h = \vec{x} - \vec{x}_p$ we have $\vec{x} = \vec{x}_p + \vec{x}_h$.

[Added after class →] ex: To solve the system $\begin{cases} x + y + 9z = 5 \\ 2x + 3y + 22z = 12 \end{cases}$ take the

augmented matrix $[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 9 & 5 \\ 2 & 3 & 22 & 12 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & 4 & 2 \end{array}\right]$

so that $\begin{cases} x \quad +5z = 3 \\ \quad y + 4z = 2 \end{cases}$ and if we let the free variable $z = t$

then solutions have the form $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - 5t \\ 2 - 4t \\ 0 + t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} = \vec{x}_p + \vec{x}_h$

where $\vec{x}_p = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is a particular solution and $\left\{ t\begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix} \right\} = \text{null}(A)$

where $A = \begin{bmatrix} 1 & 1 & 9 \\ 2 & 3 & 22 \end{bmatrix}$

Thm 4.19: The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of A.

Idea of proof: $A\vec{x} = \vec{b}$ written out is

$$A\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$ Moreover

if $A = \begin{bmatrix} \vec{c}_1 \,|\, \vec{c}_2 \,|\, \cdots \,|\, \vec{c}_n \end{bmatrix}$ where $c_i = i^{th}$ column of A

then $A\vec{x} = x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n = \vec{b}$

So the statement that there is a solution is just that $\vec{b}$ is a linear combination of columns of A

That is, $\vec{b} \in \mathrm{col}(A)$.

[Added →] ex: The statement that $100\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} + 10\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + 1\begin{bmatrix} 8 \\ 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 328 \\ 549 \\ 760 \end{bmatrix}$

is equivalent to the statement that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 10 \\ 1 \end{bmatrix}$

is a solution of the system $\begin{bmatrix} 3 & 2 & 8 \\ 5 & 4 & 9 \\ 7 & 6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 328 \\ 549 \\ 760 \end{bmatrix}$

4.7 Coordinates of a vector relative to a basis

<u>Defn</u> : We define <u>coordinates of a vector $\vec{x}$ relative to a basis B</u> as follows:

Let $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be a basis for a vector space V, and let $\vec{x}$ be a vector in V such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n .$$

The scalars $c_1, c_2, \dots c_n$ are the <u>coordinates of $\vec{x}$ relative to B</u>.

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

is the <u>coordinate vector</u> (or coordinate matrix) of $\vec{x}$ relative to B.

<u>Remark</u>: For a vector having

(1) the coordinates and
(2) the basis, allow us to

separate the ideas "How many?" and "of what?" do we have?

Added after class ↓

example: For $V = P_3 = \{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0, a_1, a_2, a_3 \in \mathbb{R} \}$

Suppose that $\vec{x} = 3 + 5t - 4t^2 + 7t^3$.

Then, relative to the standard basis for P_3

$$B = \{1, t, t^2, t^3\} \leftarrow \text{the "of what"}$$

the coordinate vector for $\vec{x}$ is

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ 5 \\ -4 \\ 7 \end{bmatrix} \leftarrow \text{the "how many"}$$

ex: It is given that the coordinate vector $[\vec{x}]_{B'}$ is

$$[\vec{x}]_{B'} = \begin{bmatrix} 5 \\ 3 \\ 2 \\ -4 \end{bmatrix}$$

$\leftarrow$ "How many"

relative to the nonstandard basis for $V = P_3$, $B' = \{ 1, 1+t, 1+t+t^2, 1+t+t^2+t^3 \}$.

"of what"

What is the vector $\vec{x}$ in $V = P_3$?

Answer: $\vec{x} = 5(1) + 3(1+t) + 2(1+t+t^2) - 4(1+t+t^2+t^3)$

$= (5+3+2-4) + (3+2-4)t + (2-4)t^2 - 4t^3$

$= 6 + 2t - 2t^2 - 4t^3 \leftarrow$ "The vector"

Remark: In a typical vector space V, the standard basis is such that it's easy to write down the coordinate vector (matrix) of a vector $\vec{x}$ in V.

ex: a) For $V = M_{2,2}$ = 2×2 matrices, the <u>standard</u> basis is

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

If $\vec{x}$ is the vector $\vec{x} = \begin{bmatrix} 12 & 5 \\ 4 & -3 \end{bmatrix}$ then the coordinate vector $[\vec{x}]_B$ relative to the standard basis is (by inspection)

$$[\vec{x}]_B = \begin{bmatrix} 12 \\ 5 \\ 4 \\ -3 \end{bmatrix}$$

b) For this <u>same vector</u> $\vec{x} = \begin{bmatrix} 12 & 5 \\ 4 & -3 \end{bmatrix}$ what is the coordinate vector $[\vec{x}]_{B'}$ relative to the <u>non-standard</u> basis $B' = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

<u>Answer</u>: we need to solve the matrix equation:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4, \quad \text{where } B' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$$

That is

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 4 & -3 \end{bmatrix}$$

This says

$$\begin{aligned} c_1 + c_2 &= 12 \\ c_3 - c_4 &= 5 \\ c_3 + c_4 &= 4 \\ c_1 - c_2 &= -3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= 9/2 = 4.5 \\ c_2 &= 15/2 = 7.5 \\ c_3 &= 9/2 = 4.5 \\ c_4 &= -1/2 = -0.5 \end{aligned}$$

so that

$$[\vec{x}]_{B'} = \begin{bmatrix} 4.5 \\ 7.5 \\ 4.5 \\ -0.5 \end{bmatrix}$$

<u>Remark</u>: This is like saying that 220 yards is the <u>same</u> distance as 660 feet: Different coordinates (220 versus 660) because they are relative to different bases (yards versus feet).