Math 254 Wed 25 July 2018

4.7 Change of basis and transition matrices

Remark: Why would you ever want different bases for a vector space?

ex: V = vector space of "proper rational functions" with denominator $(x-2)(x^2+1)$

$$= \left\{ \frac{a_0 + a_1 x + a_2 x^2}{(x-2)(x^2+1)} : a_0, a_1, a_2 \in \mathbb{R} \right\}$$

example of addition and scalar multiplication:

$$f_1(x) = \frac{3 - 2x + 5x^2}{(x-2)(x^2+1)}$$

$$f_2(x) = \frac{1 - x^2}{(x-2)(x^2+1)}$$

$$f_1(x) + f_2(x) = \frac{4 - 2x + 4x^2}{(x-2)(x^2+1)}$$

and
$$10 f_1(x) = \frac{30 - 20x + 50x^2}{(x-2)(x^2+1)}$$

standard basis

$$= B = \left\{ \frac{1}{(x-2)(x^2+1)}, \frac{x}{(x-2)(x^2+1)}, \frac{x^2}{(x-2)(x^2+1)} \right\}$$

What might be an alternative basis?

Let
$$\vec{v}_1 = \frac{1}{x-2} = \frac{x^2+1}{(x-2)(x^2+1)}$$
$$\vec{v}_2 = \frac{1}{x^2+1} = \frac{x-2}{(x-2)(x^2+1)}$$
$$\vec{v}_3 = \frac{x}{x^2+1} = \frac{x(x-2)}{(x-2)(x^2+1)}$$

and then take as a nonstandard basis (check: it is a basis)

$$B' = \{\vec{v}_1, \vec{v}_2\ \vec{v}_3\}$$

(we will come back to this.)

Theorem: (cf thm 4.20) Let $B = \{\vec{u}_1\ \vec{u}_2 \dots \vec{u}_n\}$ and let $B' = \{\vec{v}_1, v_2, \dots \vec{v}_n\}$ be two bases for a vector space V.

Definition: The matrix, denoted $P_{B \leftarrow B'}$, is

$$P_{B \leftarrow B'} = \left[\, [\vec{v}_1]_B \;\middle|\; [\vec{v}_2]_B \;\middle|\; \cdots \;\middle|\; [\vec{v}_n]_B \, \right]$$

This is the change-of-basis (or transition) matrix from B' to B.

Theorem (cont'd) For any vector $\vec{x}$ in V,

a) $$P_{B \leftarrow B'} [\vec{x}]_{B'} = [\vec{x}]_B$$

b) The change-of-basis matrix is invertible, and

$$P_{B' \leftarrow B} = \left(P_{B \leftarrow B'}\right)^{-1}$$ so that

$$P_{B' \leftarrow B} [\vec{x}]_B = \left(P_{B \leftarrow B'}\right)^{-1} [\vec{x}]_B$$
$$= [\vec{x}]_{B'}$$