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4.7 (cont'd) Change-of-basis matrices

ex: Let $V =$ proper rational function with denominator $(x-2)(x^2+1)$

$$= \left\{ \frac{a_0 + a_1 x + a_2 x^2}{(x-2)(x^2+1)} : a_0, a_1, a_2 \in \mathbb{R} \right\}$$

standard basis $= B = \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \}$

$$= \left\{ \frac{1}{(x-2)(x^2+1)}, \frac{x}{(x-2)(x^2+1)}, \frac{x^2}{(x-2)(x^2+1)} \right\}$$

non-std basis $= B' = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$

$$= \left\{ \frac{1}{x-2}, \frac{1}{x^2+1}, \frac{x}{x^2+1} \right\}$$

$$= \left\{ \frac{1+x^2}{(x-2)(x^2+1)}, \frac{-2+x}{(x-2)(x^2+1)}, \frac{-2x+x^2}{(x-2)(x^2+1)} \right\}$$

By inspection: $[\vec{v}_1]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad [\vec{v}_2]_B = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$$[\vec{v}_3]_B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

... So we can say

$$P_{B \leftarrow B'} = \left[[\vec{v}_1]_B \,\middle|\, [\vec{v}_2]_B \,\middle|\, [\vec{v}_3]_B \right]$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

How can we use this?

Consider the rational function:

$$\vec{x} = 5 \cdot \frac{1}{x-2} + 3 \frac{1}{x^2+1} + 2 \frac{x}{x^2+1}$$

$$= 5\vec{v}_1 + 3\vec{v}_2 + 2\vec{v}_3$$

That is, $[\vec{x}]_{B'} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$. What is $[\vec{x}]_B$?

$$[\vec{x}]_B = P_{B \leftarrow B'} [\vec{x}]_{B'}$$

$$= \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix}$$

That is, $\vec{x} = -1 \dfrac{1}{(x-2)(x^2+1)} - 1 \dfrac{x}{(x-2)(x^2+1)} + 7 \dfrac{x^2}{(x-2)(x^2+1)}$

$$= \frac{-1 - x + 7x^2}{(x-2)(x^2+1)}$$

We know what $P_{B \leftarrow B'}$ is. What is $P_{B' \leftarrow B}$?

$$P_{B' \leftarrow B} = \left(P_{B \leftarrow B'}\right)^{-1}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix}$$ by TI-84

It should be that

$$[\mathbb{X}]_{B'} = P_{B' \leftarrow B} [\mathbb{X}]_B$$

$$\frac{1}{5}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

So $$\mathbb{X} = \frac{-1 - x + 7x^2}{(x-2)(x^2+1)}$$

$$= \frac{5}{x-2} + \frac{3}{x^2+1} + \frac{2x}{x^2+1}$$

ex(cont'd) : What is the partial fraction decomposition of

$$f(x) = \frac{4 + 8x + 3x^2}{(x-2)(x^2+1)} \text{ ?}$$

By inspection $[f]_B = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}$

$$[f]_{B'} = P_{B' \leftarrow B} [f]_B$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 1 & 2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 6.4 \\ 1.2 \\ -3.4 \end{bmatrix} = \begin{bmatrix} 32/5 \\ 6/5 \\ -17/5 \end{bmatrix}$$

Answer: $f(x) = \frac{32}{5} \cdot \frac{1}{x-2} + \frac{6}{5} \frac{1}{x^2+1} - \frac{17}{5} \frac{x}{x^2+1}$

ex: [Another example of the use of two different bases.]

(Exercise)

Let $V = \text{Span}(\{\cos^2 x, \sin^2 x, \sin x \cos x\})$

$= \text{Span}(\{1, \cos 2x, \sin 2x\})$

That is $B = \{1, \cos 2x, \sin 2x\}$

$B' = \{\cos^2 x, \sin^2 x, \sin x \cos x\}$

what are $P_{B \leftarrow B'}$ and $P_{B' \leftarrow B}$?

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[Briefly] 5.1: Dot product in $\mathbb{R}^n$

Defn: In $\mathbb{R}^n$, if $\vec{u} = (u_1, u_2 \cdots u_n)$

and $\vec{v} = (v_1, v_2 \cdots v_n)$

the dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

ex: In $\mathbb{R}^3$ if $\vec{u} = (3, 7, 2)$

$\vec{v} = (4, 1, 5)$

then $\vec{u} \cdot \vec{v} = (3)(4) + (7)(1) + (2)(5)$

$= 12 + 7 + 10 = 29$

Defn: If $\vec{v}$ is in $\mathbb{R}^n$, we can define the length (or norm) of $\vec{v}$ by

$$\|\vec{v}\| = (\vec{v} \cdot \vec{v})^{1/2}$$

$$= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$