Math 254

Overview of Chapter 5 Tue 31 July 2018

5.1 Dot Product

ex: $\vec{v} = (3, 5, 2, 4)$ in $\mathbb{R}^4$

$\vec{w} = (2, 1, 3, 2)$

$$\vec{v} \cdot \vec{w} = (3)(2) + (5)(1) + (2)(3) + (4)(2)$$
$$= 6 + 5 + 6 + 8 = 25$$

Defn: If $\vec{v} = (v_1, v_2, v_3, \ldots v_n)$

$\vec{w} = (w_1, w_2, \ldots w_n)$

Then $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n$

Thm 5.3

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot c\vec{v}$
4. $\vec{v} \cdot \vec{v} \geq 0$ and $\vec{v} \cdot \vec{v} = 0$ only if $\vec{v} = \vec{0}$

Defn: $\|\vec{v}\|$ = length of v $= \sqrt{\vec{v} \cdot \vec{v}}$.

Note: If $\vec{v} = (v_1, v_2, v_3)$ then

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Thm 5.4 Cauchy-Schwarz Inequality

If $\vec{u}$ and $\vec{v}$ are vectors, then

$$|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$$

Remark. This says $\dfrac{|\vec{u}\cdot\vec{v}|}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$ so that

$$-1 \leq \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1$$

Defn: The angle θ between vectors $\vec{u}$ and $\vec{v}$ can be found by

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} \qquad 0 \leq \theta \leq \pi$$

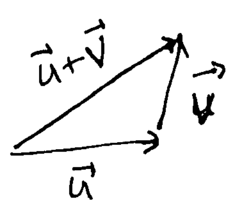
Defn: Two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are orthogonal if $\vec{u}\cdot\vec{v}=0$.

Note: This says $\cos\theta = 0$ or $\theta = \pi/2$.

Thm: 5.5 Triangle Inequality

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Reason for the name:



5.2 Inner Product Spaces

Notation: $\vec{u}\cdot\vec{v}$ = dot product in $\mathbb{R}^n$

$\langle \vec{u}, \vec{v} \rangle$ = inner product of vectors in a vector space V

Defn: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors in a vector space V, and let c be a scalar.
An inner product is a function whose input is a pair of vectors $\vec{u}$ and $\vec{v}$ and whose output is a real number satisfying:

(1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

(2) $\langle \vec{u}, \vec{v}+\vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$

(3) $c\langle \vec{u}, \vec{v} \rangle = \langle c\vec{u}, \vec{v} \rangle$

(4) $\langle \vec{v}, \vec{v} \rangle \geq 0$, and $\langle \vec{v}, \vec{v} \rangle = 0$ only if $\vec{v}=\vec{0}$

example (A nonstandard inner product in $\mathbb{R}^3$)

For $\vec{u}=(u_1, u_2, u_3)$, $\vec{v}=(v_1, v_2, v_3)$

define $\langle \vec{u}, \vec{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$

ex: (Not an inner product)

10) $\langle \vec{u}, \vec{v} \rangle = u_1 v_1 - 6 u_2 v_2$ for $\vec{u} = (u_1, u_2)$
$\vec{v} = (v_1, v_2)$

Axiom 4 fails: Let $\vec{v} = (0, 1)$

then $\langle \vec{v}, \vec{v} \rangle = (0)(0) - 6(1)(1) = -6$

ex: Let $V = C[-1, 1]$ = space of functions continuous on the interval $[-1, 1]$.

Define an inner product $\langle f, g \rangle$ as follows:
For f and g in $C[-1, 1]$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

ex: $f(x) = 2x$, $g(x) = e^{x^2}$

$$\langle f, g \rangle = \int_{-1}^{1} (2x)(e^{x^2})\, dx = \left[e^{x^2} \right]_{-1}^{1}$$

$$= e^{1^2} - e^{(-1)^2}$$

$$= e - e = 0$$

Defn: If $\langle f, g \rangle = 0$ we say that f and g are orthogonal.

<u>Defn</u>: Let $\vec{u}$ and $\vec{v}$ be vectors in an inner product space V,

(1) The <u>length</u> of u, $\|u\| = \sqrt{\langle u, u\rangle}$.

(2) The <u>distance</u> between $\vec{u}$ and $\vec{v}$ is $\|\vec{u} - \vec{v}\|$.

(3) <u>Angle</u> between $\vec{u}$ and $\vec{v}$ is found by

$$\cos\theta = \frac{\langle \vec{u}, \vec{v}\rangle}{\|\vec{u}\|\,\|\vec{v}\|}$$

<u>ex</u>: (cont'd) For $f(x) = 2x$ find $\|f\|$.

$$\langle f, f\rangle = \|f(x)\|^2 = \int_{-1}^{1} (2x)(2x)\,dx$$

$$= \int_{-1}^{1} 4x^2\,dx = \frac{4}{3}x^3\Big|_{-1}^{1}$$

$$= \frac{4}{3} - \left(-\frac{4}{3}\right) = \frac{8}{3} \quad \text{so } \|f\| = \sqrt{\frac{8}{3}} = 1.63$$

$$\frac{f(x)}{\|f(x)\|} = \frac{2x}{(8/3)^{1/2}} = \frac{\sqrt{3}}{\sqrt{8}}(2x) = \frac{\sqrt{3}}{\sqrt{2}}x = \text{unit vector.}$$

added after class

Remark: The form of the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

for $V = C[-1,1]$ should not be so surprising if one looks at the formula for the dot product:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^{3} u_i v_i$$

In $\mathbb{R}^n$, if $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots v_n)$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

If one thinks of $\vec{u} = (u_1, u_2, \ldots, u_n)$ as a function whose domain is the discrete set of integers $\{1, 2, 3, \ldots, n\}$ it resembles $f(x)$, a function whose domain is the continuous set (interval) of real numbers $[-1, 1]$. Moreover,

$$\int_{-1}^{1} f(x)\, g(x)\, dx \quad \text{resembles} \quad \sum_{i=1}^{n} u_i v_i$$

if one compares $f(x)$ with u_i, $g(x)$ with v_i, dx with i, and the integral symbol with the summation symbol.