Math 254 Wed 1 Aug 2018 (1)

Warmup questions

(1) Let $\vec{u}_1 = 1$, $\vec{u}_2 = \sin t$, $\vec{u}_3 = \cos t$,
and let $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. As a subspace of $C(-\infty,\infty)$,
Let $V = \text{span}(S)$.
Show that S is linearly independent, so that S is a basis for V, and $\dim(V) = 3$.

Answer: Consider the equation $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = \vec{0}$, that is,

$$(\star) \qquad c_1 1 + c_2 \sin t + c_3 \cos t = 0$$

Let $t=0$: $\quad c_1 1 + c_2 \sin 0 + c_3 \cos 0 = 0$ or

(1): $c_1 + c_3 = 0$

Let $t = \frac{\pi}{2}$ (2): $c_1 + c_2 = 0$

Let $t = \pi$ (3): $c_1 - c_3 = 0$

(1)+(3): $2c_1 = 0 \Rightarrow \boxed{c_1 = 0}$

Sub into (2): $\boxed{c_2 = 0}$

Sub into (1): $\boxed{c_3 = 0}$

$\therefore$ The only solution of the equation $(\star)$ is the trivial solution,
That is, S is linearly independent.

M254 18.08.01

Warmup question

(2) Let $\vec{v_1} = \ln t^2$, $\vec{v_2} = 2 + \ln t$, $\vec{v_3} = 1$

be elements of the vector space $C(0,\infty)$, and let

$V = \text{span}(S)$ where $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$.

Show that S is linearly dependent.

What is $\dim V$?

Answer: Consider the equation

$$(\star) \quad c_1 \ln(t^2) + c_2(2 + \ln t) + c_3 1 = 0$$

Because $\ln(t^2) = 2\ln t$,

$$2c_1 \ln t + 2c_2 + c_2 \ln t + c_3 = 0$$

Collecting like terms:

$$(2c_2 + c_3) + (2c_1 + c_2)\ln t = 0$$

This equation is satisfied if the following system is satisfied:

$$\begin{cases} \quad 2c_2 + c_3 = 0 \\ 2c_1 + c_2 \quad = 0 \end{cases}$$ with coefficient matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/2 \end{bmatrix}$$ with solutions $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} t/4 \\ -t/2 \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Take $t = 4$ so that $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ is one nontrivial solution of $(\star)$:

$$\ln t^2 - 2(2 + \ln t) + 4 = 0$$

So that S is linearly dependent.

Warmup question (2) (cont'd):

Claim: $\dim(V) = 2$

Reason: Because $\vec{v}_1 = 2\vec{v}_2 - 4\vec{v}_3$, that is,

$$\ln t^2 = 2(2+\ln t) - 4\cdot 1 \in \operatorname{span}(\{\vec{v}_2, \vec{v}_3\})$$

$$V = \operatorname{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}) = \operatorname{span}(\{\vec{v}_2, \vec{v}_3\})$$

Moreover, $\{\vec{v}_2, \vec{v}_3\}$ is linearly independent, as neither of

$\vec{v}_2 = 2 + \ln t$ nor $\vec{v}_3 = 1$

are scalar multiples of each other.

$\therefore$ $\{\vec{v}_2, \vec{v}_3\}$ is a basis for V, hence

$\dim(V) = 2$.

<u>A bit</u> about 5.3 Orthogonal bases and the Gram-Schmidt process.

<u>Defn</u>: A set of vectors ~~is~~ $S = \{v_1\ v_2 \cdots v_n\}$ is <u>orthogonal</u> if $\langle v_i, v_j \rangle = 0$ for $i \neq j$.

A orthogonal set of vectors is <u>orthonormal</u> if $\langle v_i, v_i \rangle = 1$ for each i.