Math 254 2 Aug 2018

6.1 Intro Linear Transformations

Remark. Three situations where we use matrices:

(1) Column matrices: to represent coordinates of a vector relative to a basis.

e.g. In P_2, $p(x) = 3 + 5x - x^2$

$$[p]_B = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$ relative to $B = \{1, x, x^2\}$.

(2) Change-of-basis matrix (a.k.a "transition matrix"):

$$P_{B \leftarrow B'} = \left[[v_1]_B \,\middle|\, [v_2]_B \,\middle|\, \cdots \,\middle|\, [v_n]_B \right]$$

where $B' = \{\vec{v}_1, \vec{v}_2 .. \vec{v}_n\}$

NOTE: A change-of-basis matrix is always square and invertible.

new → (3) We will use $m \times n$ matrices to represent a "linear transformation" from a vector space V with $\dim(V) = n$ to a vector space W with $\dim(W) = m$.

V = domain of the linear transformation T

W = codomain "

Defn: Let V and W be vectorspaces.

The function $T : V \to W$

is a linear transformation of V into W when the following properties are true for all $\vec{u}$ and $\vec{v}$ in V and for any scalar:

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$.

ex: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(x_1, x_2) = (x_1 + x_2, -2x_1 + 2x_2)$$

check: Is (1) true? Let $\vec{u} = (x_1, x_2)$ $\vec{v} = (y_1, y_2)$

$$T(\vec{u}+\vec{v}) = T(x_1+y_1, x_2+y_2)$$

$$= ((x_1+y_1) + (x_2+y_2), -2(x_1+y_1) + 2(x_2+y_2))$$

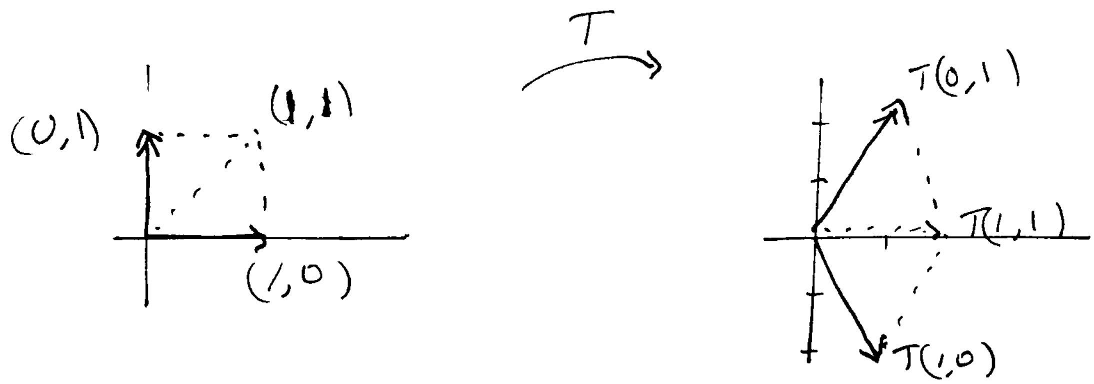
equal? Yes.

on the other hand $T(\vec{u}) + T(\vec{v}) = T(x_1, x_2) + T(y_1, y_2)$

$$= (x_1 + x_2, -2x_1 + 2x_2) + (y_1 + y_2, -2y_1 + 2y_2)$$

One can also check: $T(c\vec{u}) = cT(\vec{u})$

ex (cont'd) : How to visualize this example



Note

$$T(1,0) = (1+0,\ -2\cdot 1 + 2\cdot 0) = (1,-2)$$

$$T(0,1) = (0+1,\ -2\cdot 0 + 2\cdot 1) = (1,\ 2)$$

So $T(1,1) = (1+1,\ -2\cdot 1 + 2\cdot 1) = (2,\ 0)$

ex: 6.1 #10) Why is $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x,y) = (x, y^2)$$

NOT a linear transformation?

Counterexample to the claim that T is a linear transformation:

$$T(0,1) = (0, 1^2) = (0,1)$$

$$T(10(0,1)) = T(0,10) = (0,100)$$ Not equal

$$\neq 10\ T(0,1) = 10\ (0,1) = (0,10)$$

That is, $T(c\vec{u}) \neq c\,T(\vec{u})$

Theorem 6.1: The following will be true for any linear transformation $T: V \to W$.

1. $T(\vec{0}) = \vec{0}$
2. $T(-\vec{v}) = -T(\vec{v})$
3. $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$
4. If $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n v_n$

then $T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(v_2) + \dots + c_n T(v_n)$

In words: "The image of a linear combination of vectors is the linear combination of the images of the vectors."

Remark: Because of 4., a linear transformation is completely determined by its effect of a basis for V.

Thm 6.2: Let A be an $m \times n$ matrix. The function $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\vec{v}) = A\vec{v}$$

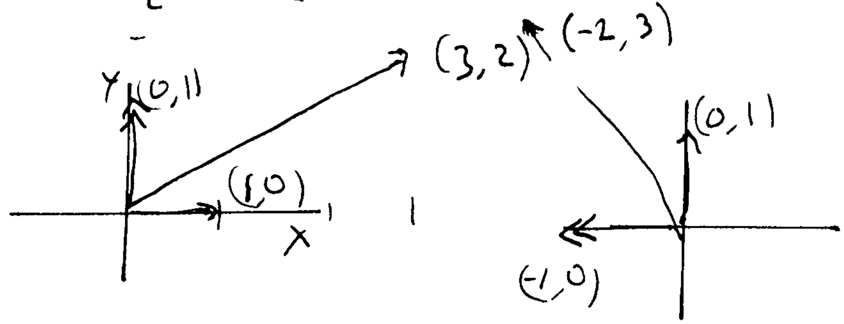
is a linear transformation.

ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ define $T: \mathbb{R}^2 \to \mathbb{R}^2$

by $T(x, y) = A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

So $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$



So $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ i.e this T rotates counterclockwise 90°.

<u>Remark</u>: The converse of this theorem is sort of true.
For a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ there is always a matrix A, called the "standard matrix of the linear transformation T"

<u>ex</u>: $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$T(1,0,0) = (5,3)$
$T(0,1,0) = (7,1)$
$T(0,0,1) = (2,4)$

and then extend to all of $\mathbb{R}^3$ by linearity.

"Standard matrix of T"

$$A = \begin{bmatrix} 5 & 7 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

$\uparrow$ $T(100)$ $\quad \uparrow$ $T(010)$ $\quad \uparrow$ $T(001)$

So if you have some arbitrary vector in $\mathbb{R}^3$,

like $(1, 2, 3)$, you want to find $T(1, 2, 3)$

just do

$$A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}5 & 7 & 2\\3 & 1 & 4\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}25\\17\end{bmatrix}$$

__ex__: $T: P^3 \to P^2$ (an example of a linear transformation on vector spaces other than $\mathbb{R}^n$ and $\mathbb{R}^m$.)

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$$

Aha! The derivitive operator is actually a linear transformation.

6.2 The Kernel and Range of Linear Transformation

__Defn__: Let $T: V \to W$ be a linear transformation.

The set of vectors $\vec{v}$ in the domain V

that satisfy $T(\vec{v}) = \vec{0} \in W$

is called __kernel of T__, denoted $\ker(T)$.

__Thm 6.3__: The kernal of T is a __subspace__ of the domain V.

__proof__: $\ker(T)$ is nonempty for $T(\vec{0}) = \vec{0}$. Also, $\ker(T)$ is closed under addition: If $T(\vec{v}_1) = \vec{0}$ and $T(\vec{v}_2) = 0$, then $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$. Finally, T is closed under scalar multiplication: If $T(\vec{v}) = 0$ then $T(c\vec{v}) = cT(\vec{v}) = c\vec{0} = \vec{0}$.

ex: Find the kernel, ker(T), for

$T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by left multiplication

by $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$

That is, solve $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

That is, find the nullspace of A.

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 9/2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -17/2 \\ 0 & 1 & 9/2 \end{bmatrix}$$

Solve $\begin{cases} x - \frac{17}{2} z = 0 \\ y + \frac{9}{2} z = 0 \end{cases}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{17}{2} t \\ -\frac{9}{2} t \\ t \end{bmatrix} = t \begin{bmatrix} 17/2 \\ -9/2 \\ 1 \end{bmatrix}$$

so $\ker(T) = \left\{ t\left(\frac{17}{2}, \frac{-9}{2}, 1\right) : t \in \mathbb{R} \right\}$