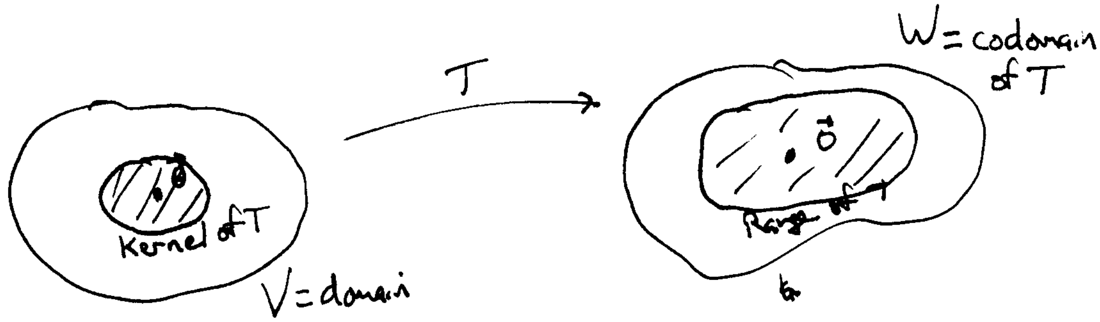
Math 254 Mon 6 Aug 2018

6.2 (cont'd) Range of a linear transformation

<u>Defn</u>: If $T: V \to W$ is a linear transformation

$$\text{range}(T) = \{T(\vec{v}) : \vec{v} \text{ in } V\}$$

<u>Thm 6.4</u>: The range, range(T), of a linear transformation $T: V \to W$ is a subspace of W.



<u>proof of 6.4</u>: [not hard]

<u>Corollaries</u>: (1) If $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$T(\vec{x}) = A\vec{x}$$

then $\ker(T) = \text{nullspace}(A)$.

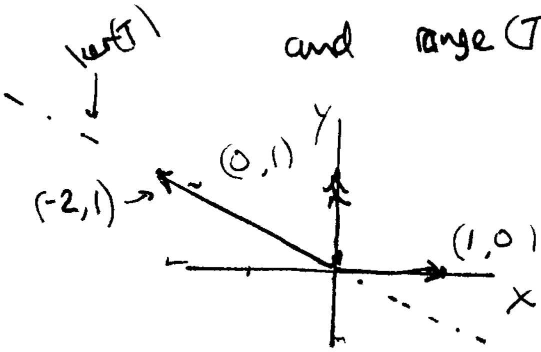
(2) ... then $\text{range}(T) = \text{col}(A)$.

ex: Define $T : \mathbb{R}^2 \to \mathbb{R}^2$

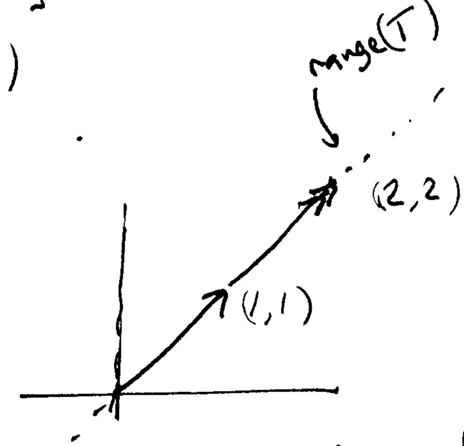
by $T(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$

where $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

Find $\ker(T) = \text{nullspace}(A)$
and $\text{range}(T) = \text{col}(A)$



$\xrightarrow{T}$



$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

<u>Find range(T)</u> = col(A)

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\text{col}(A) = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \text{ in } \mathbb{R} \right\} \Rightarrow \dim(\text{range}(T)) = \text{rank}(A) = 1$$

That is, range(T) is a 1-dimensional subspace of $\mathbb{R}^2$.

ex (cont'd)

Find $\ker(T) = \text{nullspace}(A) = N(A)$: Solve $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \quad x + 2y = 0$ so y is free, let $y = t$ then $x = -2t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\ker(T) = \left\{ t\begin{bmatrix} -2 \\ 1 \end{bmatrix} : t \text{ in } \mathbb{R} \right\}$$

a 1 dimensional subspace of the domain of T.

$$\dim(\ker(T)) = \dim(\text{nullspace}(A)) = \text{nullity}(A) = 1$$

<u>Defn</u>: Let $T: V \to W$ be a linear transformation.

$\dim(\ker(T)) = \text{nullity}(T)$ is the <u>nullity of T</u>.

$\dim(\text{range}(T)) = \text{rank}(T)$ is the <u>rank of T</u>.

<u>Thm 6.5</u> "Sum of Rank and Nullity"

If $T: V \to W$ is a linear transformation with $\dim(V) = n$, then

$$\text{rank}(T) + \text{nullity}(T) = n$$

That is

$$\dim(\text{range}) + \dim(\text{kernel}) = \dim(\text{domain}).$$

<u>Essence of proof</u>:

number of col's with leading 1s + number of col's without leading 1s = total number of columns of A

<u>Thm 6.6</u> : "One-to-one Linear Transformation"

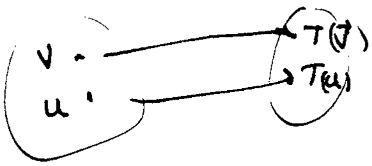
Let $T: V \to W$ be a linear transformation.

Then T is one-to-one if and only if

$\ker(T) = \{\vec{0}\}$. <u>proof</u> (not hard)

<u>Recall</u> : T is one-to-one if

$\vec{u} \neq \vec{v} \Rightarrow T(\vec{u}) \neq T(\vec{v})$ for any $\vec{u}, \vec{v}$ in V



<u>Equivalent</u> : is the contrapositive statement.

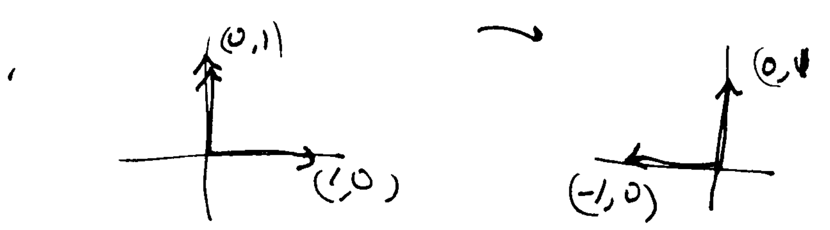
$$T(\vec{u}) = T(v) \Rightarrow \vec{u} = \vec{v}$$

<u>Previous example</u>: $\ker(T) = \text{span}\left(\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}\right) \neq \{\vec{0}\}$
= not trivial

$\Rightarrow$ T is NOT one-to-one.

<u>ex</u> : $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\ker(T) = \text{nullspace}(A) = \{\vec{0}\}$ so T is one-to-one



Thm 6.7: "Onto L.Ts."

Let $T: V \to W$ where W is finite dimensional. Then T is onto if and only if

$$\text{rank}(T) = \dim(W).$$

~~ex~~ Recall: $T: V \to W$ is onto if every $\vec{w}$ in W has the property that $\vec{w} = T(\vec{v})$ for some $\vec{v}$ in V.

ex (2) T defined by $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not onto.

ex (1): T defined by $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is onto.