7.1 Eigenvalues and eigenvectors

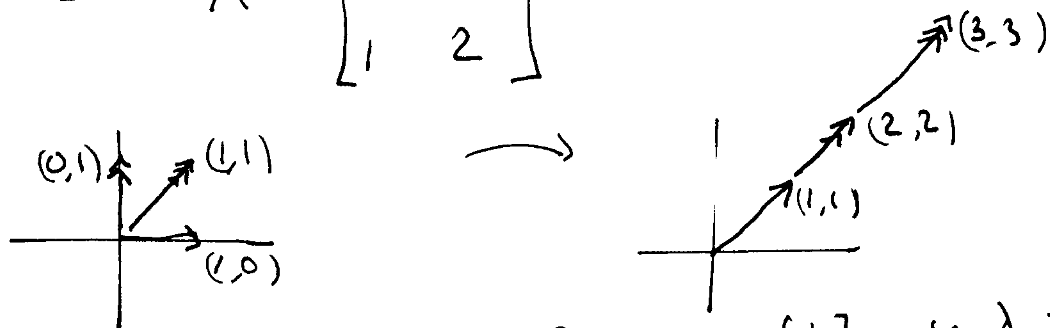
<u>Remark</u>: For a square matrix A it sometimes happens that $A\vec{x}$ is a scalar multiple of $\vec{x}$, for some vector $\vec{x}$.

<u>Defn</u>: Let A be an $n \times n$ matrix.
The scalar λ is an <u>eigenvalue</u> of A when there is a <u>nonzero</u> vector $\vec{x}$ such that:

$$A\vec{x} = \lambda \vec{x}$$

The vector $\vec{x}$ is called an eigenvector of A corresponding to λ.

<u>ex</u>: $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$



$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $\lambda_1 = 3$ is an <u>eigenvalue</u> of A with corresponding <u>eigenvector</u> $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Also $$A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So $\lambda_2 = 0$ is an <u>eigenvalue</u> with eigenvector $\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Remark: One can easily test that a vector $\vec{x}$ is an eigenvector, and if "yes" what the eigenvalue would be.

Remark: A different task: Given an eigenvalue λ of A how can we find a corresponding eigenvector.

ex: #9) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$ has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$.

Find corresponding eigenvectors.

TRICK: Given that λ is an eigenvalue:

That means: $A\vec{x} = \lambda \vec{x}$ for some $\vec{x} \neq \vec{0}$.

$$A\vec{x} = \lambda I \vec{x} \Rightarrow \vec{0} = \lambda I \vec{x} - A\vec{x}$$

$$\Rightarrow \vec{0} = (\lambda I - A)\vec{x}$$

That is, we just need to find a vector in null $(\lambda I - A)$.

ex(cont'd): $\lambda_1 I - A = -1 I - A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

Solve $\begin{bmatrix} -5 & 5 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\xrightarrow{\text{rref}}$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has soln $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So we may take $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to $\lambda_1 = -1$.

ex (cont'd) $\lambda_2 = 2$ is also an eigenvalue (it's given)

Find a corresponding eigenvector.

$$\vec{0} = (\lambda_2 I - A)\vec{x} = \left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -5/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 5/2 \\ 1 \end{bmatrix}$

If $t=2$, $\vec{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda_2 = 2$.

check: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$A\left[\begin{array}{c|c} 1 & 5 \\ 1 & 2 \end{array}\right] = \left[\begin{array}{c|c} -1 & 10 \\ -1 & 4 \end{array}\right] = \left[\begin{array}{c|c} 1 & 5 \\ 1 & 2 \end{array}\right]\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

(A slightly premature) Observation: If we let $P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}$ so that $P^{-1} = \begin{bmatrix} -2/3 & +5/3 \\ +1/3 & -1/3 \end{bmatrix}$

$$AP = P\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = P\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$ multiply on left by P^{-1}:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$ ← we say that A is diagonalizable if $P^{-1}AP = D$ where D is a diagonal matrix.

Remark: Hey can we avoid having someone tell us what the eigenvalues λ_1, λ_2 etc are?

Idea: λ is an eigenvalue of A if $A\vec{x}=\lambda\vec{x}$, for $\vec{x}\neq 0$.

Equivalent: $(\lambda I - A)\vec{x} = \vec{0}$ for $\vec{x}\neq 0$.

Equivalent (by Great equivalence theorem): $\det(\lambda I - A) = 0$.

"Characteristic polynomial"

16) Find the characteristic equation for $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

and find its solutions (the eigenvalues of A).

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda - 1 & 4 \\ 2 & \lambda - 8 \end{bmatrix}$$

$$\Rightarrow 0 = |\lambda I - A| = (\lambda-1)(\lambda-8) - (2)(4)$$
$$= \lambda^2 - 9\lambda + 8 - 8 = \lambda^2 - 9\lambda$$
$$= \lambda(\lambda - 9)$$

So the eigenvalues are $\boxed{\lambda_1 = 0}$, $\boxed{\lambda_2 = 9}$.

↓[Added after class] Here are the corresponding eigenvectors:

For $\lambda_1 = 0$,

$$\lambda_1 I - A = \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ So we may take

$$\vec{x}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$ is a corresponding eigenvector

For $\lambda_2 = 9$

$$\lambda_2 I - A = \begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -\frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$ If we take $t=2$, then

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$ is a corresponding eigenvector.

Theorem 7.1 "Eigenvectors of λ Form a Subspace"

If A is an $n \times n$ matrix with an eigenvector λ, then the set of all eigenvectors of λ, together with the zero vector.

$$\{\vec{x} : \vec{x} \text{ is an eigenvector of } \lambda\} \cup \{\vec{0}\}$$
$$= \{\vec{x} : A\vec{x} = \lambda\vec{x}\}$$

is a subspace of $\mathbb{R}^n$.

proof: [Closed under scalar multiplication]: If $\vec{x}$ is an eigenvector of A such that $A\vec{x} = \lambda\vec{x}$, then $c\vec{x}$ also an eigenvector for $A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x})$.

[Closed under addition] If $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors of A corresponding to the same eigenvalue λ, then so is $\vec{x}_1 + \vec{x}_2$, for $A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \lambda\vec{x}_1 + \lambda\vec{x}_2 = \lambda(\vec{x}_1 + \vec{x}_2)$. $\square$

ex (7.1 # 70) Find the dimension, and a basis, for the eigenvalue $\lambda = 3$ if $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, given that $\lambda = 3$ is an eigenvalue of A.

Answer: $\lambda I - A = 3I - A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The equation $(3I - A)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then satisfies $x_2 = 0$ but $x_1 = s$ and $x_3 = t$ are free variables.

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This describes a 2-dimensional subspace of $\mathbb{R}^3$ with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

That is, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent eigenvectors of $\lambda = 3$.