Math 141 Wed 18 FEB 2015

3.3 Long division

ex: We have 47 wooden blocks. How many stacks of 3 can we make?

$$47 = 3 + 44 = 3 + 3 + 41 = 3 + 3 + 3 + 38$$

$$= 3 \cdot 10 + 17$$

$$= 3 \cdot 10 + 3 \cdot 5 + 2 = 3 \cdot 15 + 2$$

conclusion: $$\underset{\text{dividend}}{\underset{\uparrow}{47}} = \underset{\text{divisor}}{\underset{\uparrow}{3}} \cdot \underset{\text{quotient}}{\underset{\uparrow}{15}} + \underset{\text{remainder}}{\underset{\nwarrow}{2}}$$

OR: $$\frac{47}{3} = 15 + \frac{2}{3}$$

$$\begin{array}{r} 15 \text{ r. } 2 \\ 3\overline{)\,47\phantom{\text{ r. } 2}} \\ \underline{30}\phantom{\text{ r. } 2} \\ 17\phantom{\text{ r. } 2} \\ \underline{15}\phantom{\text{ r. } 2} \\ 2\phantom{\text{ r. } 2} \end{array}$$

Analogous: $(3x^3+x^2+4x+2)\div(x-2)$

$$\begin{array}{rl} & 3x^2+7x+18 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow \quad x-2\,\big) & \overline{3x^3+x^2+4x+2} \leftarrow \text{dividend} \\ & 3x^3-6x^2 \\ & \overline{\quad 7x^2+4x+2} \\ & \quad 7x^2-14x \\ & \overline{\qquad 18x+2} \\ & \qquad 18x-36 \\ & \overline{\qquad\quad 38} \leftarrow \text{remainder} \end{array}$$

<u>NOTE</u>: $1-(-6)=7$
$4-(-14)=18$
$2-(-36)=38$

<u>Conclusion</u>:

$$\text{dividend} \rightarrow P(x)=3x^3+x^2+4x+2 = \underset{\nearrow\ \text{divisor}}{(x-2)}\,(3x^2+\underset{\uparrow\ \text{quotient}}{7x}+18) + \underset{\nwarrow\ \text{remainder}}{38}$$

<u>OR</u>:

$$\frac{3x^3+x^2+4x+2}{x-2} = 3x^2+7x+18+\frac{38}{x-2}$$

<u>NOTE</u>: $P(2) = \cancelto{0}{(2-2)}\cdot(3\cdot 2^2+7\cdot 2+18)+38 = 38$

<u>Remainder Theorem</u>: If the polynomial $P(x)$ is divided by $x-c$ then the remainder is $P(c)$.

<u>Factor Theorem</u>: c is a <u>zero</u> of $P(x)$ if and only if $(x-c)$ is a <u>factor</u> of $P(x)$.

<u>Reason</u>: Special case of the remainder theorem when the remainder is 0.

Horner's algorithm

ex.: $P(x) = (3x^3 + x^2 + 4x) + 2$

$$= (3x^2 + x + 4)x + 2$$

$$= [(3x^2 + x) + 4]x + 2$$

$$= [(3x + 1)x + 4]x + 2$$

Evaluate P(2).

$$P(2) = [(3 \cdot 2 + 1) \cdot 2 + 4] \cdot 2 + 2$$

$$= [7 \cdot 2 + 4] \cdot 2 + 2$$

$$= 18 \cdot 2 + 2$$

$$= 38$$

Synthetic division by $x - c$ ex.: Divide by $x - 2$

$$\begin{array}{r|rrrr} 2 & 3 & 1 & 4 & 2 \\ & & 6 & 14 & 36 \\ \hline & 3 & 7 & 18 & \big] \; 38 = P(2) \end{array}$$

$$P(x) = (x-2)(3x^2 + 7x + 18) + 38$$

Remark: ① we add in synthetic division, in contrast with long division, where we subtract.

② In both long and synthetic division, if there is a missing term, we write 0 as a placeholder to make the algorithm work.

3.3 46) Use {Synthetic division/ Horner's algorithm to evaluate $P(-2)$

where $P(x) = 6x^5 + 10x^3 + x + 1$

divide by $x+2$ ↘

$$\begin{array}{r|rrrrrl} -2 & 6 & 0 & 10 & 0 & 1 & 1 \\ & & -12 & 24 & -68 & 136 & -274 \\ \hline & 6 & -12 & 34 & -68 & 137 & \big| -273 = P(-2) \end{array}$$

By-product: $\dfrac{P(x)}{x+2} = 6x^4 - 12x^3 + 34x^2 - 68x + 137 - \dfrac{273}{x+2}$

58) $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6$

Confirm that $1/3$ and -2 are zeros, then find the other two zeros.

$$\begin{array}{r|rrrrl} -2 & 3 & -1 & -21 & -11 & 6 \\ & & -6 & 14 & 14 & -6 \\ \hline \frac{1}{3} & 3 & -7 & -7 & 3 & \big| \; 0 = P(-2) \\ & & 1 & -2 & -3 & \\ \hline & 3 & -6 & -9 & \big| \; 0 & \end{array}$$

$$\begin{aligned} P(x) &= (x+2)(3x^3 - 7x^2 - 7x + 3) \\ &= (x+2)\left(x - \tfrac{1}{3}\right)(3x^2 - 6x - 9) = (x+2)\left(x - \tfrac{1}{3}\right) \cdot 3(x^2 - 2x - 3) \\ &= (x+2)(3x-1)(x^2 - 2x - 3) = (x+2)(3x-1)(x-3)(x+1) \end{aligned}$$

$$\text{zeros: } \quad \boxed{-2} \quad \boxed{\tfrac{1}{3}} \quad \boxed{3} \quad \boxed{-1}$$

3.4 Rational zeros theorem

If $P(x) = a_n x^n + \cdots + a_1 x + a_0$ has <u>integer</u> coefficients then any <u>rational</u> zero has the form $\frac{p}{q}$ where p divides evenly into a_0 and q divides evenly into a_n.

<u>ex</u>: Consider a polynomial with zeros $\frac{1}{3}$ and $\frac{2}{5}$. like

$$P(x) = 3\cdot\left(x-\tfrac{1}{3}\right)\cdot 5\left(x-\tfrac{2}{5}\right) = (3x-1)(5x-2)$$

$$\text{zeros: } \quad \updownarrow \tfrac{1}{3} \qquad \updownarrow \tfrac{2}{5}$$

multiplied out:

$$P(x) = \underset{\uparrow}{15}x^2 - 11x + \underset{\uparrow}{2}$$

3 and 5 divide evenly into 15

1 and 2 divide evenly into 2.

<u>How we will use this</u>: Given $P(x) = 15x^2 - 11x + 2$ ask what are the <u>possible</u> rational zeros?

Possible values of p: $\pm 1, \pm 2$

" of q: $\pm 1, \pm 3 \ \pm 5, \pm 15$

" of $\frac{p}{q}$: $\pm 1, \pm 2, \pm\frac{1}{3} \ \pm\frac{2}{3}, \pm\frac{1}{5}, \pm\frac{2}{5} \ \pm\frac{1}{15} \ \pm\frac{2}{15}$

<u>Idea</u>: If there are <u>any</u> rational zeros, they appear in this list of sixteen numbers.

Find the rational zeros of, then factor,

3.4 30) $P(x) = 2x^4 - x^3 - 19x^2 + 9x + 9$

Possible rational zeros: $p: \pm 1, 3, 9$

$q: \pm 1, 2$

$\frac{p}{q}: \pm 1, \frac{1}{2}, 3, \frac{3}{2}, 9, \frac{9}{2}$ ← Twelve

Is -1 a zero?

$$\begin{array}{r|rrrr|r} -1 & 2 & -1 & -19 & 9 & 9 \\ & & -2 & 3 & 16 & -25 \\ \hline & 2 & -3 & -16 & 25 & -16 \end{array} = P(-1)$$

Nope. $(x+1)$ is not a factor.

Is 3 a zero?

$$\begin{array}{r|rrrr|r} 3 & 2 & -1 & -19 & 9 & 9 \\ & & 6 & 15 & -12 & -9 \\ \hline 1 & 2 & 5 & -4 & -3 & 0 \\ & & 2 & 7 & 3 & \\ \hline -3 & 2 & 7 & 3 & 0 & \\ & & -6 & -3 & & \\ \hline & 2 & 1 & 0 & & \end{array}$$

$0 = P(3)$ Yup. $(x-3)$ is a factor

Is 1 a zero of the remaining factor?

Is -3 a zero of this latest factor?

$$\begin{aligned} P(x) &= (x-3)(2x^3+5x^2-4x-3) \\ &= (x-3)(x-1)(2x^2+7x+3) \\ &= (x-3)(x-1)(x+3)(2x+1) \end{aligned}$$

$$\begin{array}{lcccc} & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \text{Zeros:} & 3 & 1 & -3 & -\frac{1}{2} \end{array}$$

3.4 67) $P(x) = 2x^6 + 5x^4 \underset{\frown}{-} x^3 - 5x - 1$

one variation, so 1 positive (real) zero.

$P(-x) = 2(-x)^6 + 5(-x)^4 - (-x)^3 - 5(-x) - 1$

$= 2x^6 + 5x^4 + x^3 + 5x \underset{\frown}{-} 1$

one variation, so there is 1 negative zero.

Zeros? $\underbrace{\text{one positive, one negative}}_{\text{two real}}$, then $\underbrace{\text{four complex zeros.}}_{\text{four complex}}$

$\underbrace{\text{two real} \qquad \text{four complex}}_{\text{six zeros}}$

80) Find the zeros of $P(x) = 2x^4 + 15x^3 + 31x^2 + 20x + 4$

Descartes' Rule of Signs: no variations, so no positive zeros.

$P(-x) = 2x^4 - 15x^3 + 31x^2 - 20x + 4$

4 variations; 4 or 2 or no negative zeros.

Rational zeros? $\left.\begin{array}{l} p: \pm 1, \pm 2 \quad \pm 4 \\ q: \pm 1, \pm 2 \end{array}\right\}$ $\frac{p}{q}: \begin{array}{lll} \pm 1 & \pm 2 & \pm 4 \\ \pm \frac{1}{2} & \cancel{\pm \frac{2}{2}} & \cancel{\pm \frac{4}{2}} \end{array}$

only consider: $-1, -2, -4, -\frac{1}{2}$.

By use of a TI-83, these appear to be x-intercepts of $y = P(x)$. →

$$\begin{array}{r|rrrrr} -2 & 2 & 15 & 31 & 20 & 4 \\ & & -4 & -22 & -18 & -4 \\ \hline -\frac{1}{2} & 2 & 11 & 9 & 2 & \vert\ 0 \\ & & -1 & -5 & -2 & \\ \hline & 2 & 10 & 4 & \vert\ 0 & \end{array}$$

← To find the last two zeros, use the quadratic formula to solve $x^2 + 5x + 2 = 0$ to get $x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 2}}{2(2)} = \frac{-5 \pm \sqrt{17}}{4}$

(Added after class ended.)

$P(x) = (x+2)(x+\frac{1}{2})(2x^2 + 10x + 4) = (x+2)(2x+1)(x^2+5x+2)$

$= (x+2)(2x+1)(x + \frac{5-\sqrt{17}}{2})(x + \frac{5+\sqrt{17}}{2})$

zeros: $-2 \qquad -\frac{1}{2} \qquad \frac{-5+\sqrt{17}}{2} \qquad \frac{-5-\sqrt{17}}{2}$