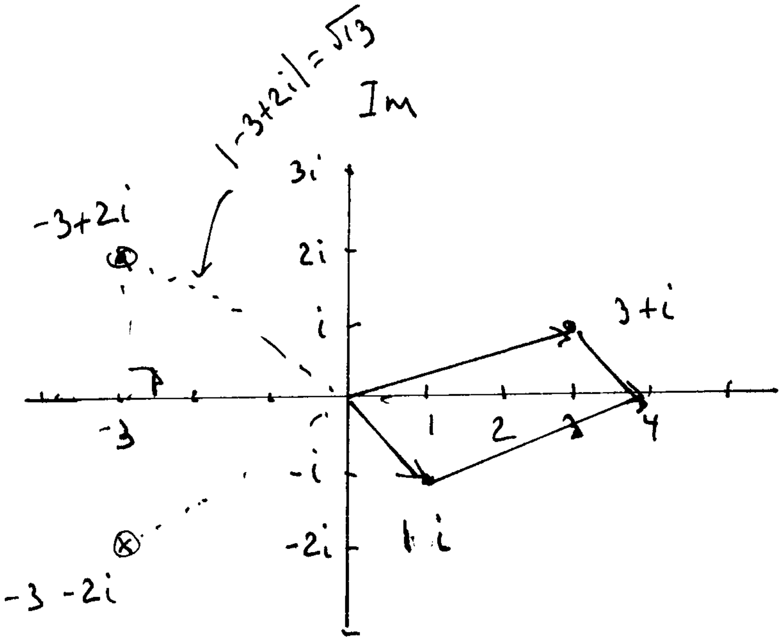
3.6 (cont'd) How to visualize the Complex numbers



$(3+i)+(1-i)$
$= 4$

ex: Write a polynomial with real coefficient which has $-3+2i$ as a zero.

Since zeros will appear as conjugate pairs, if $-3+2i$ is a zero, then $-3-2i$ is also a zero.

$$\begin{aligned}
P(x) &= [x-(-3+2i)]\,[x-(-3-2i)] \\
&= (x+3-2i)(x+3+2i) \\
&= [(x+3)-2i]\,[(x+3)+2i] \\
&= (x+3)^2-(2i)^2 \\
&= (x^2+6x+9)-(4i^2) \\
&= x^2+6x+9+4
\end{aligned}$$

$$\boxed{P(x) = x^2+6x+13}$$

has zeros $-3 \pm 2i$

NOTE: This polynomial is irreducible over the real numbers.

more
<u>generally</u>: If $z = a+bi$ is a zero of real polynomial then so is $\bar{z} = a-bi$ and a resulting irreducible quadratic factor will be

$$[x - (a+bi)]\,[x - (a-bi)]$$
$$= [(x-a) - bi]\,[(x-a)+bi]$$
$$= (x-a)^2 - b^2 i^2$$
$$= x - 2ax + a^2 + b^2$$
$$= x - 2ax + (a^2+b^2) \leftarrow |z|^2 \text{ if } z = a+bi$$

$2a = 2 \cdot$ real part of z,

<u>Defn</u>: For a complex number $z = a+bi$ the <u>absolute</u> <u>value</u> of z,

$$|z| = \sqrt{a^2+b^2} = (z \cdot \bar{z})^{1/2}$$

<u>Fact</u>: A polynomial with real coefficients always factor into either linear factors, or irreducible quadratic factors corresponding complex conjugate zeros.

<u>ex</u>: Show that $P(x) = x^4 + 4$ factors.

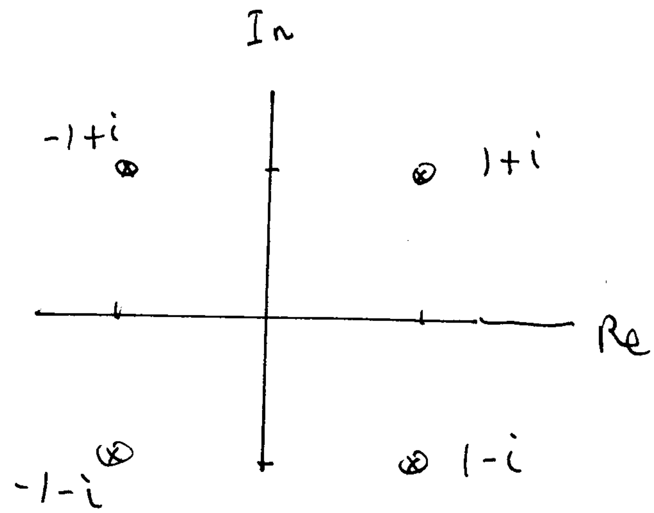
<u>TRICK</u>: $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2$

$$= (x^2+2)^2 - (2x)^2$$
$$= [(x^2+2) - (2x)]\,[(x^2+2) + (2x)]$$
$$= (x^2 - 2x + 2)\,(x^2 + 2x + 2)$$
$$\Updownarrow \qquad\qquad \Updownarrow$$

zeros: $1 \pm i$ $\qquad -1 \pm i$

zeros of

$P(x) = x^4 + 4$



ex (like a quiz problem): Find a polynomial of least possible degree with zeros $\frac{1}{2}$ and $5i$.

solution: Given that we're looking for a polynomial with real coefficients, since $5i$ is a zero, $-5i$ must also be a zero. One possible answer:

$$P(x) = (2x-1)(x-5i)(x+5i)$$

zeros: $\quad \frac{1}{2} \qquad 5i \qquad -5i$

$$= (2x-1)(x^2 - 25i^2)$$

$$= (2x-1)(x^2+25)$$

$$P(x) = 2x^3 - x^2 + 50x - 25$$

ex [quiz-like problem]

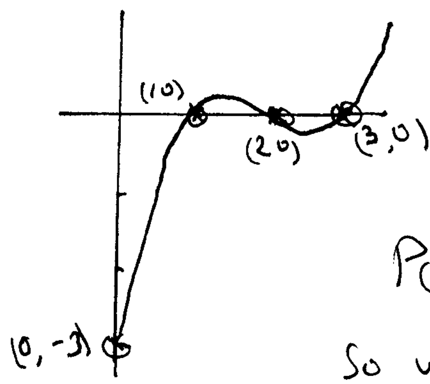
write $(1+2i)^3$ in the form $a+bi$

Recall: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$(1+2i)^3 = 1^3 + 3(1)^2(2i) + 3(1)(2i)^2 + (2i)^3$$
$$= 1 + 6i \quad -12 \quad -8i$$
$$= -11 - 2i$$

ex.



Find the equation of this cubic polynomial function.

$$P(x) = a(x-1)(x-2)(x-3)$$

So what is a ?

$$-3 = P(0) = a(0-1)(0-2)(0-3)$$
$$-3 = -6a \Rightarrow a = \frac{-3}{-6} = \frac{1}{2}$$

$$P(x) = \frac{1}{2}(x-1)(x-2)(x-3)$$
$$= \frac{1}{2}(x-1)(x^2-5x+6)$$
$$= \frac{1}{2}(x^3 - 6x^2 + 11x - 6)$$
$$= \frac{1}{2}x^3 - 3x^2 + \frac{11}{2}x - 3$$

Scratch

$$\begin{array}{r} x^2 - 5x + 6 \\ x - 1 \\ \hline -x^2 + 5x - 6 \\ x^3 - 5x^2 + 6x \quad \\ \hline x^3 - 6x^2 + 11x - 6 \end{array}$$

3.6 25) Factor, find zeros, state multiplicity.

$$P(x) = 16x^4 - 81 = (4x^2)^2 - 9^2$$
$$= (4x^2-9)(4x^2+9)$$
$$= \left[(2x)^2 - 3^2\right](4x^2+9)$$
$$= (2x-3)(2x+3)(4x^2+9)$$

zeros: $\frac{3}{2}$ $\quad -\frac{3}{2}$ $\quad ?$ $\quad ?$

Solve: $4x^2 + 9 = 0$
$$4x^2 = -9$$
$$x^2 = -\frac{9}{4}$$
$$x = \pm\sqrt{\frac{-9}{4}} = \pm \frac{3}{2} i$$

zeros: $\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}i, -\frac{3}{2}i$ each of multiplicity 1.

53) $P(x) = x^4 + x^3 + 7x^2 + 9x - 18$

no. of positive zeros: 1

$$P(-x) = x^4 - x^3 + 7x^2 - 9x - 18$$

no. of negative zeros: 3 or 1

Rational zeros: $\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 6 \quad \pm 9 \quad \pm 18$

$$\begin{array}{r|rrrrr}
1 & 1 & 1 & 7 & 9 & -18 \\
  &   & 1 & 2 & 9 & 18 \\
\hline
-2 & 1 & 2 & 9 & 18 & 0 \\
   &   & -2 & 0 & -18 & \\
\hline
   & 1 & 0 & 9 & 0 &
\end{array}$$

so $P(x) = (x-1)(x^3+2x^2+9x+18)$
$$= (x-1)(x+2)(x^2+9)$$

zeros: $1, -2, 3i, -3i$

Solve $x^2 + 9 = 0$
$$x^2 = -9 \Rightarrow x = \pm\sqrt{-9} = \pm 3i$$

3.7 Rational functions $\quad r(x) = \dfrac{P(x)}{Q(x)}$

Prealgebra Facts: $\dfrac{0}{q} = 0$, $\dfrac{p}{0}$ is undefined

$\dfrac{1}{BIG} = \text{small}$, $\dfrac{1}{\text{small}} = BIG$

$(\text{neg})(\text{neg}) = \dfrac{(\text{neg})}{(\text{neg})} = \text{pos}$, $(\text{neg})(\text{pos}) = \dfrac{(\text{neg})}{(\text{pos})} = \text{neg}$

Example 5: $r(x) = \dfrac{2x^2+7x-4}{x^2+x-2}$

- $r(0) = \dfrac{-4}{-2} = 2$ so $(0,2) =$ y-intercept
- $r(1000) = r(BIG) = \dfrac{2(BIG)^2 + \cdots}{(BIG)^2 + \cdots} \approx \dfrac{2(BIG)^2}{1(BIG)^2} = \dfrac{2}{1} = 2$

so the line $y=2$ is a horizontal asymptote

Now, factor: $r(x) = \dfrac{(2x-1)(x+4)}{(x-1)(x+2)}$ ← zeros: $\frac{1}{2}, -4$ ← zeros: $1, -2$

- x-intercepts $(\frac{1}{2}, 0)$ $(-4, 0)$
- vertical asymptotes: $x = 1$, $x = -2$