4.3 and 4.4 } Logs and Properties of Logs

Defn of $\log_a$:

$$\boxed{y = \log_a x \quad \text{means} \quad a^y = x}$$

ex: $\underbrace{5}_{\text{exponent}} = \log_{\underbrace{2}_{\text{base}}} \overbrace{32}^{\text{other}}$ means $\underbrace{2}_{\text{base}}^{\overbrace{5}^{\text{exponent}}} = 32 \leftarrow$ other

ex: Write in equivalent log form $10^4 = 10{,}000$

log form: $\log_{10}(10{,}000) = 4$

ex: write equivalent exponential form

$\log_3 \frac{1}{81} = -4$ ← Both true.

Equivalent: $3^{-4} = \frac{1}{81}$ ←

ex: Find $\log_4 \frac{1}{8}$. Let $x = \log_4 \frac{1}{8}$ and solve for x.

Equivalent: $4^x = \frac{1}{8}$

$(2^2)^x = 2^{-3}$

$2^{2x} = 2^{-3}$

since → $y = 2^x$ is one-to-one:

$2x = -3$

$x = \boxed{-\frac{3}{2}}$

Six
Properties of logs

(1) $\log_a a^x = x$
(2) $a^{\log_a x} = x$
Inverse properties

(3) $\log_a (AB) = \log_a A + \log_a B$ Product rule.

(4) $\log_a \left(\frac{A}{B}\right) = \log_a A - \log_a B$ Quotient rule.

(5) $\log_a (A^C) = C \log_a A$ Power rule.

(6) $\log_b x = \dfrac{\log_a x}{\log_a b}$ Change-of-base formula.

ex of (1): $\log_2 2^5 = 5$ $\quad \log_3 3^{-4} = -4$

$\log_{10} 10^{1/2} = \frac{1}{2}$

special cases: $\log_7 7 = \log_7 7^1 = 1$

$\log_{15} 1 = \log_{15} 15^0 = 0$

ex's of (2): $10^{\log_{10} 1.5} = 1.5$ $\quad 2^{\log_2 (x^2+1)} = x^2+1$

$e^{\log_e 17} = 17$

Examples of 3, 4 and 5

ex: $\log_{10}[(100)\cdot(1000)] = \log_{10}(100) + \log_{10}(1000)$

That is: $5 = 2 + 3$

ex: $\log_{10}\left(\frac{100{,}000}{1000}\right) = \log_{10}(100{,}000) - \log_{10}(1000)$

That is: $2 = 5 - 3$

ex: $\log_{10} 100^3 = 3 \log_{10} 100$

That is: $6 = 3\cdot(2)$

Examples of 6: $\log_8 4 = \frac{\log_2 4}{\log_2 8} = \frac{\log_2 2^2}{\log_2 2^3} \overset{\text{by } 1}{=} \frac{2}{3}$

(and major shortcut)

change-of-base
with $x=4$
$b=8$,
$a=2$

ex: $\log_{81}\sqrt{3} = \log_{81} 3^{1/2} = \frac{\log_3 3^{1/2}}{\log_3 81} = \frac{\log_3 3^{1/2}}{\log_3 3^4} = \frac{\left(\frac{1}{2}\right)}{4}$

$= \frac{1}{2} \div 4 = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$

why is (1) true? why is $\log_a a^x = x$ true?

Well, $\underset{\text{base}}{a}^{\overset{\text{exponent}}{x}} = \underbrace{(a^x)}_{\text{other}}$ is true.

Equivalent log form: $\log_{\underset{\text{base}}{a}} \overset{\text{other}}{(a^x)} = \overset{\text{exponent}}{x}$

why is (2) true?

$\log_{\underset{\text{base}}{a}} \overset{\text{other}}{x} = \underbrace{(\log_a x)}_{\text{exponent}}$

equivalent exponential eqn: $a^{\log_a x} = x$

why is (3) true? $\log_a(AB) = \log_a A + \log_a B$

[understand this just in case it comes up on a test]

Let $x = \log_a A$ and $y = \log_a B$

Equivalent exp. form: $a^x = A$ and $a^y = B$

Observe $\underset{\text{other}}{AB} = a^x \cdot a^y \underset{\substack{\text{essence of}\\ \text{this derivation}}}{=} \underset{\text{base}}{a}^{\overset{\text{exp.}}{x+y}}$

Equivalent log form: $\log_a(AB) = x + y$

back-substituting: $\log_a(AB) = \log_a A + \log_a B$

Review: 1) $\ln x$ means $\log_e x$

$\ln x$ is called the natural log of x.

2) $\log x$ means $\log_{10} x$

$\log x$ is called the common log of x.

Remarks: In very advanced math classes
$\log x$ may often mean $\ln x$.

In computer science courses
$\log x$ may often ~~mean~~ $\log_2 x$.

ex: What is $\log_3 27$? $\log_3 27 = \log_3 3^3 = 3$.

ex: What is $\log_3 25$? A bit less than 3.

observe: $3^2 = 9 < 25 < 27 = 3^3$

$2 = \log_3 3^2 < \log_3 25 < \log_3 3^3 = 3$

We can do better: Let $\overset{\text{exp}}{x} = \log_{\underset{\text{base}}{3}} \overset{\text{other}}{25}$

Equivalent: $3^x = 25$

TRICK: $\ln 3^x = \ln 25$

by (5): $x \ln 3 = \ln 25$

so $x = \dfrac{\ln 25}{\ln 3} = \dfrac{3.21888}{1.0986} = 2.929947041$

≈ 2.93

[Imitate the previous example:]

Derivation of (6): Find $\log_b A$ (where b = mystery base, a = familiar base)

Let $x = \log_b A$

Equivalent: $b^x = A$

$$\log_a b^x = \log_a A$$

By (5) power rule: $x \log_a b = \log_a A$

$$x = \frac{\log_a A}{\log_a b}$$

Back-substitute: $\boxed{\log_b A = \frac{\log_a A}{\log_a b}}$

Remark: In practice we will be taking $a = e$ or $a = 10$.

Exercises §4.4

30) Expand $\log_a \frac{x^2}{yz^3} = \log_a x^2 - \log_a yz^3$ by (4)

$= \log_a x^2 - (\log_a y + \log_a z^3)$ by (3)

$= \log_a x^2 - \log_a y - \log_a z^3$ distribute the minus.

$= 2\log_a x - \log_a y - 3\log_a z$

44) Expand

$$\log\left(\frac{10^x}{x(x^2+1)(x^4+2)}\right) = \underbrace{\underbrace{\log 10^x}_{\log_{10} 10^x}}_{x} - \log[x(x^2+1)(x^4+2)]$$

$= x - [\log x + \log(x^2+1) + \log(x^4+2)]$

$= x - \log x - \log(x^2+1) - \log(x^4+2)$

Another way to look at 30) in §4.4.

Expand $\log_a \frac{x^2}{yz^3} = \log_a (x^2 \cdot y^{-1} \cdot z^{-3})$

$= \log_a x^2 + \log_a y^{-1} + \log_a z^{-3}$

$= 2 \log_a x + (-1) \cdot \log_a y + (-3) \log_a z$

$= 2 \log_a x - \log_a y - 3 \log_a z$

4.5 Exp and Log eqns

<u>Exp</u> eqns: a common base is possible.

<u>ex</u>. $2^x = 8$

$2^x = 2^3$

$x = 3$

<u>ex</u>: $2^{x^2-6x} = \frac{1}{512}$

$2^{x^2-6x} = 2^{-9}$

$x^2 - 6x = -9$

$x^2 - 6x + 9 = 0$

$(x-3)^2 = 0$

$x - 3 = 0 \Rightarrow x = 3$

<u>ex</u>: $9^{x+1} = 27^{x-1}$

$(3^2)^{x+1} = (3^3)^{x-1}$

$3^{2x+2} = 3^{3x-3}$

$2x + 2 = 3x - 3$
$-2x \quad\quad -2x$

$2 = x - 3$
$+3 \quad\quad +3$

$5 = x$

Exp. eqns.: In the case in which there is No hope for a common base

ex: $2^{x+3} = 7$

$\ln 2^{x+3} = \ln 7$ (common log would also work)

by the Power Rule (5) $(x+3)\ln 2 = \ln 7$

$$\frac{(x+3)\cancel{\ln 2}}{\cancel{\ln 2}} = \frac{\ln 7}{\ln 2}$$

$$x+3 = \frac{\ln 7}{\ln 2}$$

$$x = -3 + \frac{\ln 7}{\ln 2}$$ ← Exact solution

$= -0.192645$ ← Approximate solution

ex: Solve $2500 = 1000\, e^{0.12t}$

$$\frac{2500}{1000} = \frac{\cancel{1000}\, e^{0.12t}}{\cancel{1000}}$$

$$2.5 = e^{0.12t}$$

$$\ln 2.5 = \ln e^{0.12t}$$ ← $\log_e e^{0.12t} = 0.12t$ by (7)

$$\ln 2.5 = 0.12t$$

$$\frac{\ln 2.5}{0.12} = \frac{\cancel{0.12}t}{\cancel{0.12}}$$

$$\frac{\ln 2.5}{0.12} = t$$

$$t \approx \frac{0.91629}{0.12} = 7.64$$

4.5 30) $e^{2x} - e^{x} - 6 = 0$

Behold: $(e^{x})^2 - e^{x} - 6 = 0$ Let $u = e^{x}$.

$u^2 - u - 6 = 0$

$(u+2)(u-3) = 0$

$u+2=0$ OR $u-3=0$

$u=-2$ OR $u=3$

$e^{x} \neq -2$ OR $e^{x}=3$

No solution because e^{x} is always positive

$\ln e^{x} = \ln 3$

$\boxed{x = \ln 3}$

(Begin) Log equations

ex $\log_3(x-5) = 2$ log form

$3^{\log_3(x-5)} = 3^2$

$x-5 = 3^2$ equivalent Exp. form

$x-5=9$

$x=14$

ex: $\log_5(x+1) = \log_5 7$

$x+1=7$

$x=6$