warmup

7.1 p.499 78) Verify the identity

$$\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = 2\sec x \tan x$$

$$\text{LHS} = \frac{1}{(1-\sin x)}\cdot\frac{(1+\sin x)}{(1+\sin x)} - \frac{1}{(1+\sin x)}\cdot\frac{(1-\sin x)}{(1-\sin x)}$$

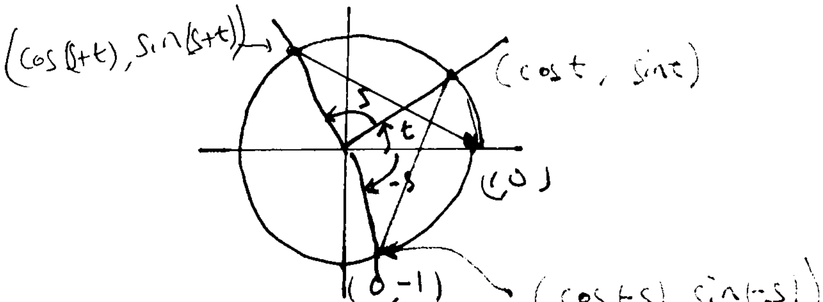
$$= \frac{1+\sin x - 1 + \sin x}{1-\sin^2 x} \underset{\text{pythagorean ID}}{=} \frac{2\sin x}{\cos^2 x}$$

$$= 2\cdot\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \underset{\text{Quotient ID}}{=} 2\sec x \tan x = \text{RHS}$$

7.2 Addition and Subtraction formulas
Start with one hard identity:

$$\boxed{\cos(s+t) = \cos s \cos t - \sin s \sin t}$$

Sketch of the proof: $s + t = t - (-s)$



Equate the length^2 of the two chords.

$$[\cos(s+t) - 1]^2 + [\sin(s+t) - 0]^2 = [\cos t - \underbrace{\cos(-s)}_{\cos s}]^2 + [\sin t - \underbrace{\sin(-s)}_{-\sin s}]^2 \quad \text{[even/odd ID]}$$

$$\cos^2(s+t) - 2\cos(s+t) + 1 + \sin^2(s+t) = \cos^2 t - 2\cos t\cos s + \cos^2 s + \sin^2 t + 2\sin t\sin s + \sin^2 s$$

pythagorean ID

$$2 - 2\cos(s+t) = 2 - 2[\cos t\cos s - \sin t \sin s]$$

$$\cos(s+t) = \cos s \cos t - \sin s \cdot \sin t$$

__Remark__: __All__ the remaining identities are (relatively) easy to derive.

__ex__: Derive the difference formula:

$$\cos(s-t) = \cos(s+(-t))$$
$$= \cos s \cos(-t) - \sin s \sin(-t)$$
$$= \cos s \cos t + \sin s \sin t$$

$$\boxed{\cos(s-t) = \cos s \cos t + \sin s \sin t}$$

__ex__: Derive a formula for $\sin(s+t)$.

$$\sin(s+t) = \cos\left(\tfrac{\pi}{2} - (s+t)\right) = \cos\left[\left(\tfrac{\pi}{2} - s\right) - t\right]$$
$$= \cos\left(\tfrac{\pi}{2} - s\right)\cos t + \sin\left(\tfrac{\pi}{2} - s\right)\sin t$$
$$= \sin s \cdot \cos t + \cos s \cdot \sin t$$

so $$\boxed{\sin(s+t) = \sin s \cos t + \cos s \sin t}$$

__Also__

$$\boxed{\begin{aligned} \tan(s+t) &= \frac{\tan s + \tan t}{1 - \tan s \tan t} \\ \tan(s-t) &= \frac{\tan s - \tan t}{1 + \tan s \tan t} \end{aligned}}$$

7.2 20) Find the exact value of

$$\cos\frac{13\pi}{15}\cos\left(-\frac{\pi}{5}\right)-\sin\frac{13\pi}{15}\sin\left(-\frac{\pi}{5}\right)$$

$$=\cos\left[\frac{13\pi}{15}+\left(-\frac{\pi}{5}\right)\right]=\cos\left(\frac{13\pi}{15}-\frac{3\pi}{15}\right)$$

$$=\cos\left(\frac{10\pi}{15}\right)=\cos\frac{2\pi}{3}=-\cos\frac{\pi}{3}=\boxed{-\frac{1}{2}}$$

because $\frac{2\pi}{3}$ is in QII and cosine is negative there.

34) Verify the identity:

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y$$

$$\begin{aligned}LHS=\cos(x+y)+\cos(x-y)&=\cos x\cos y-\sin x\sin y\\&\quad+\cos x\cos y+\sin x\sin y\\&=2\cos x\cos y\end{aligned}$$

$= RHS$

7.3 Double Angle, Half angle etc.

$$\boxed{\begin{aligned}&(1)\quad \sin 2x=2\sin x\cos x\\&(2)\quad \cos 2x=\cos^2 x-\sin^2 x\\&(3)\quad =2\cos^2 x-1\\&(4)\quad =1-2\sin^2 x\\&(5)\quad \tan 2x=\frac{2\tan x}{1-\tan^2 x}\end{aligned}}$$

Solve (3) for $\cos^2 x$:

$$\cos 2x = 2\cos^2 x - 1$$

$$1+\cos 2x = 2\cos^2 x$$

$$\boxed{\frac{1+\cos 2x}{2}=\cos^2 x}$$

Formula for Lowering Power

Likewise, use (4):

$$\boxed{\frac{1-\cos 2x}{2}=\sin^2 x}$$

And also

7.3 12) Use power lowering IDs to rewrite $\cos^4 x$ to remove powers of cosine.

$$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1+2\cos 2x+\cos^2 2x}{4}$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2(2x)$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left[\frac{1+\cos 2(2x)}{2}\right]$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8}\left[1 + \cos 4x\right]$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$

$$\boxed{\cos^4 x = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x}$$

← an example of a "trigonometric polynomial"

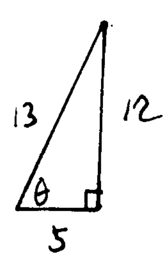
48) Find the exact value of $\cos\left(2\tan^{-1}\frac{12}{5}\right)$.

Let $\theta = \tan^{-1}\frac{12}{5}$ so that $\tan\theta = \frac{12}{5}$.

We are looking for $\cos 2\theta = 2\cos^2\theta - 1$.

What is $\cos\theta$?

$\tan\theta = \frac{opp}{adj} = \frac{12}{5}$



So $\cos\theta = \frac{adj}{hyp} = \frac{5}{13}$ and

So $hyp = \sqrt{5^2+12^2}$

$= \sqrt{25+144}$

$= \sqrt{169} = 13$

$$\cos 2\theta = 2(\cos\theta)^2 - 1$$

$$= 2\left(\frac{5}{13}\right)^2 - 1$$

$$= 2\left(\frac{25}{169}\right) - 1 = \frac{50}{169} - \frac{169}{169}$$

$$= \boxed{-\frac{119}{169}}$$

Half angle identities

Start with the Power Lowering ID:

example:

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

and let $x = \frac{u}{2}$

so that $2x = u$.

$$\cos^2 \frac{u}{2} = \frac{1+\cos u}{2}$$

Take square roots:

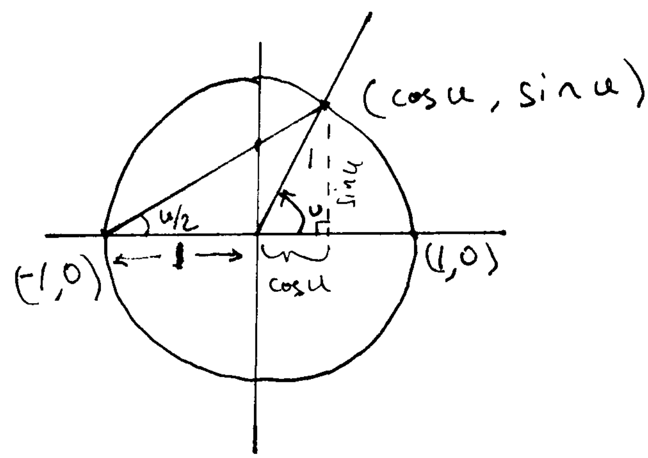
$$\boxed{\cos \frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}}$$

Similarly

$$\boxed{\sin \frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}}$$

$$\boxed{\tan \frac{u}{2} = \frac{\sin u}{1+\cos u} = \frac{1-\cos u}{\sin u}}$$

Graphical interpretation of the first half angle identity for tangent:



$$\tan \frac{u}{2} = \frac{\text{opp}}{\text{adj}} = \frac{\sin u}{1+\cos u}$$