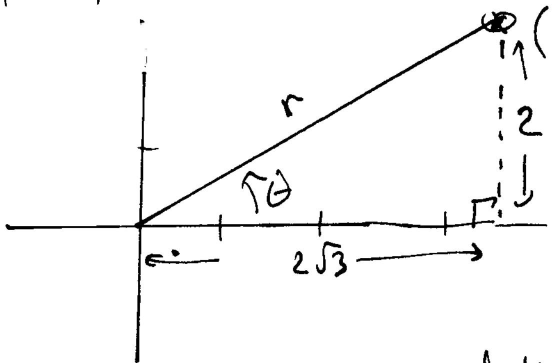
8.1 Polar coordinates



A different set of two numbers to describe this location.

r = distance from the origin

$$r^2 = (2\sqrt{3})^2 + 2^2 = 12 + 4 = 16$$

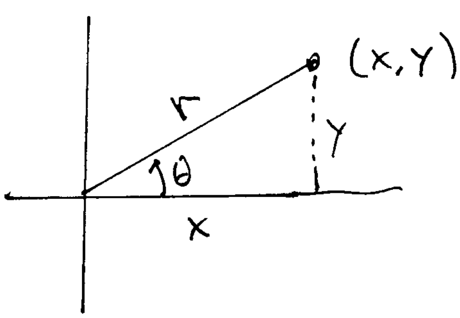
$$r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = \boxed{4}$$

θ = counter-clockwise angle about the origin

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$$

more generally:

Rectangular coordinates (x, y)
Polar coordinates (r, θ)



Because $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$$\boxed{\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}}$$

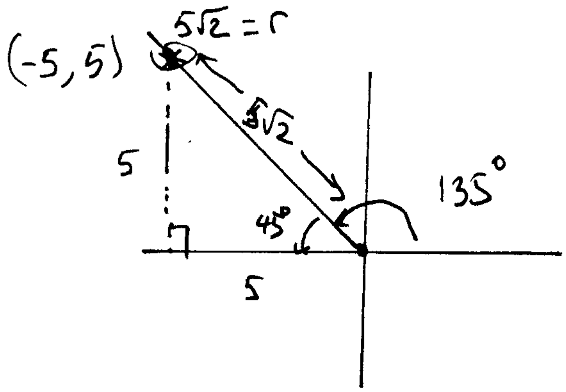
Because

$$x^2 = r^2\cos^2\theta$$
$$y^2 = r^2\sin^2\theta$$

$$x^2 + y^2 = r^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1}$$

Because $\frac{y}{x} = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \tan\theta$

ex: Suppose that the polar coordinates of a point are
$r = 5\sqrt{2}$ and $\theta = 135^\circ$.
what are (x,y) = rectangular coordinates?



← Answer this geometrically,
OR use the identities

$x = r\cos\theta = 5\sqrt{2}\cos 135^\circ = 5\sqrt{2}\cdot\left(-\frac{\sqrt{2}}{2}\right) = -5$

$y = r\sin\theta = 5\sqrt{2}\sin 135^\circ = 5\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 5$

Exercises §8.1 #28, #40.

Remark: ① In rectangular coordinates there is a one-to-one correspondence between ordered ~~pairs~~ pairs and points.

② In polar coordinates, one point corresponds to many values of θ and r.

example: $(x,y) = (-5,5)$ corresponds to $(r,\theta) = (5\sqrt{2}, 135^\circ)$

But also $(r,\theta) = (5\sqrt{2}, 495^\circ)$
OR $(r,\theta) = (5\sqrt{2}, -225^\circ)$

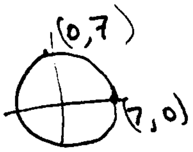
There's more: $(r,\theta) = (-5\sqrt{2}, -45^\circ)$
or $(r,\theta) = (-5\sqrt{2}, 315^\circ)$

check: $x = r\cos\theta = -5\sqrt{2}\cos(-45^\circ) = (-5\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) = -5$

$y = r\sin\theta = -5\sqrt{2}\sin(-45^\circ)\ (-5\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) = 5$

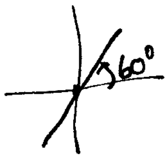
$(x,y) = (-5,5)$

ex [Converting equations Polar $\leftrightarrow$ Rectangular]



1) $x^2+y^2=49$ Because $r^2=x^2+y^2$

This is equivalent to $r^2=49$ or $r=7$

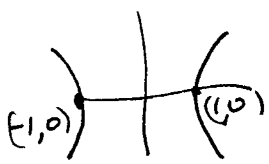


2) $y=\sqrt{3}\,x$ $\quad \frac{y}{x}=\sqrt{3}$ $\quad$ Since $\frac{y}{x}=\tan\theta$

$\tan\theta=\sqrt{3}$ or

$\theta=60^\circ=\frac{\pi}{3}$

8.1 #48) $x^2-y^2=1$ express in terms of r and θ



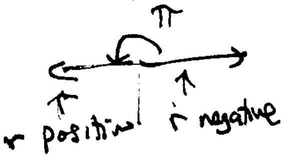
A hyperbola

$$(r\cos\theta)^2-(r\sin\theta)^2=1$$
$$r^2\cos^2\theta-r^2\sin^2\theta=1$$
$$r^2(\cos^2\theta-\sin^2\theta)=1$$
$$r^2\cos 2\theta=1$$
$$r^2=\frac{1}{\cos 2\theta}$$
$$\boxed{r^2=\sec 2\theta}$$

53) $r\cos\theta=6$ Express in terms of x and y.

In general $x=r\cos\theta$
$y=r\sin\theta$

$\boxed{x=6}$



52) $\theta=\pi$

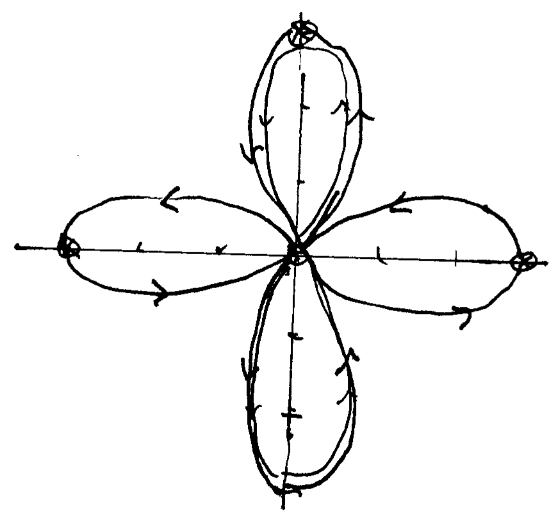
$\tan\theta=\tan\pi=0$

$\frac{y}{x}=0 \Rightarrow \boxed{y=0}$

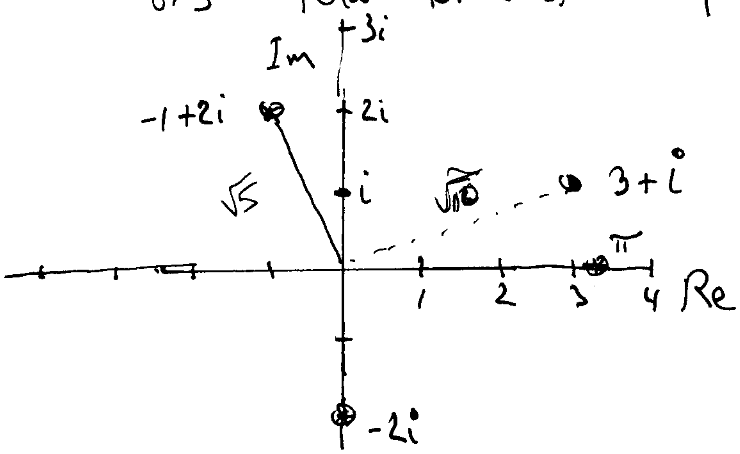
8.2 Graphs of polar equations

ex: $r = 3\cos 2\theta$

θ	r
0	$3\cos 2(0) = 3$
$\pi/4$	$3\cos 2\cdot\frac{\pi}{4} = 3\cos\frac{\pi}{2} = 0$
$\pi/2$	$3\cos 2\cdot\frac{\pi}{2} = 3(-1) = -3$
$3\pi/4$ ~~$3\pi/2$~~	0
π	3
$5\pi/4$	0
$3\pi/2$	-3
$7\pi/4$	0
2π	3



8.3 Polar form of Complex Numbers (analog of the number line for real numbers)



Defn A complex number written as $a+bi$ is in standard form or rectangular form.

Defn: The absolute value or modulus of $z = a+bi$ is $|z| = |a+bi|$
$= \sqrt{a^2+b^2}$
= distance from origin

ex: $|3+i| = \sqrt{3^2+1^2} = \sqrt{10}$

$|-1+2i| = \sqrt{(-1)^2+2^2} = \sqrt{5}$

$|-7+0i| = \sqrt{(-7)^2+0^2} = 7$

The polar form of a complex number $z = a+bi$

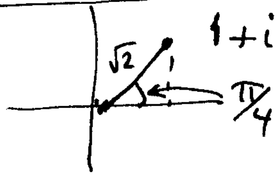
$$z = r(\cos\theta + i\sin\theta)$$

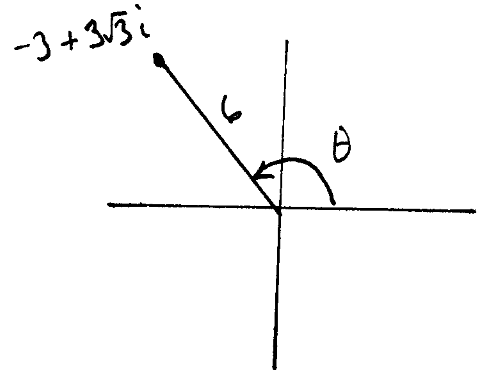
where $r = |z| = \sqrt{a^2+b^2}$ = modulus of z

and $\tan\theta = \frac{b}{a}$ where θ is called the argument of z.

ex: If $z = \boxed{1+i}$ standard form, $r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$

polar form $\boxed{z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$ $\tan\theta = \frac{b}{a} = \frac{1}{1} = 1$ so $\theta = \frac{\pi}{4}$





ex: $z = \boxed{-3 + 3\sqrt{3}\,i}$ standard form

$r = \sqrt{(-3)^2 + (3\sqrt{3})^2}$

$= \sqrt{9+27} = \sqrt{36} = 6$

$\tan\theta = \frac{b}{a} = \frac{3\sqrt{3}}{-3} = -\sqrt{3}$

Since we want the Quadrant II solution, take $\theta = \frac{2\pi}{3} = 120°$

Polar form of z: $\boxed{z = 6\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)}$ polar form

ex: In polar form $z = 5\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

what is the standard form for z.

$z = 5\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 5\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right)$

$= (5\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) + (5\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right)i = -5 - 5i$

For complex number..

Remark: 1 adding is easy using standard form

2 multiplying is easy using polar form.

~~In words~~ If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $z_1 z_2 = r_1 r_2 \left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$$

ex: $z_1 = 12\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_2 = 3\left(\cos\frac{\pi}{6} + i\sin\pi/6\right)$

$$z_1 z_2 = \cancel{36}\ 12\cdot 3\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4}+\pi/6\right)\right]$$

$$= 36\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \frac{12}{3}\left[\cos\left(\frac{\pi}{4}-\pi/6\right) + i\sin\left(\frac{\pi}{4}-\pi/6\right)\right]$$

$$= 4\left(\cos\frac{\pi}{12} + i\sin\pi/12\right)$$

In words: To multiply z_1 and z_2

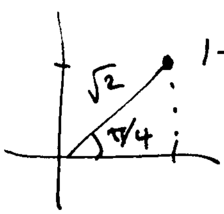
1 multiply the moduli and 2 add the arguments

To divide

1 divide the moduli and 2 subtract the arguments

ex: $z = 1 + i$

a) Find the polar (or trigometric form) of z.

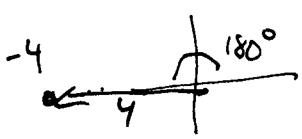


$$z = \boxed{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$$
$$= \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}$$

b) Find $z^4 = z \cdot z \cdot z \cdot z$

$$z^4 = \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2}\left[\cos\left(\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{4}+\frac{\pi}{4}\right)\right]$$
$$= \sqrt{2}^4\left[\cos\left(4\cdot\frac{\pi}{4}\right) + i\sin\left(4\cdot\frac{\pi}{4}\right)\right]$$
$$= \boxed{4\left(\cos\pi + i\sin\pi\right)}$$

c) write z in standard form.



$$z^4 = 4\left(\cos\pi + i\sin\pi\right)$$
$$= 4(-1 + 0i) = -4 + 0i$$
$$= \boxed{-4}$$

Afterclass Remark: If we wanted to find a 4th root of -4, we could have done these steps in reverse:

$$-4 \xrightarrow[\text{polar}]{\text{convert to}} 4(\cos\pi + i\sin\pi) \xrightarrow[\pi \div 4 = \frac{\pi}{4}]{4^{1/4} = \sqrt{2}} \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\xrightarrow[\text{to standard form}]{\text{convert}} 1 + i$$