9.2 Dot product (briefly)

ex: $\vec{v} = \langle 2, 5\rangle \qquad \vec{w} = \langle 4, 7\rangle$

$$\vec{v}\cdot\vec{w} = (2)(4) + (5)(7) = 8 + 35 = 43$$

ex: $\vec{v} = \langle 2, 5\rangle \qquad \vec{u} = \langle 5, -2\rangle$

$$\vec{v}\cdot\vec{u} = (2)(5) + (5)(-2) = 10 - 10 = 0$$

ex: $\vec{v} = \langle 2, 5\rangle \qquad \vec{j} = \langle 0, 1\rangle$

$$\vec{v}\cdot\vec{j} = (2)(0) + (5)(1) = 5$$

Defn: $\vec{u} = \langle a_1, b_1\rangle$ and $\vec{v} = \langle a_2, b_2\rangle$

define the dot product of $\vec{u}$ and $\vec{v}$

$$\boxed{\vec{u}\cdot\vec{v} = a_1 a_2 + b_1 b_2}$$

← a number not a vector.

Facts:
1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2. $(a\vec{u})\cdot\vec{v} = \vec{u}\cdot a\vec{v} = a(\vec{u}\cdot\vec{v})$
3. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
4. $\vec{u}\cdot\vec{u} = |\vec{u}|^2$

ex: $\vec{u} = \langle 3, 4\rangle \qquad \vec{u}\cdot\vec{u} = (3)(3) + (4)(4) = 9 + 16 = 25$

$|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25}$ so $|u|^2 = 25 = \vec{u}\cdot\vec{u}$

<u>ex</u>: Since $\vec{i} = \langle 1, 0 \rangle$ $\vec{j} = \langle 0, 1 \rangle$

$$\vec{i}\cdot\vec{i} = 1 \qquad \vec{j}\cdot\vec{j} = 1 \qquad \vec{i}\cdot\vec{j} = 0$$

we can the vector $\vec{i}$ and $\vec{j}$ form an <u>orthonormal</u> <u>system</u> because

(1) $\vec{i}\cdot\vec{j} = 0$ says $\vec{i}$ and $\vec{j}$ are orthogonal

(2) $|\vec{i}| = 1$ and $|\vec{j}| = 1$ both have length 1.

~~Fact: Geometric~~

<u>Remark</u>: These three facts, together with the distributive property completely determine the dot product.

<u>ex</u>: $\vec{u} = 2\vec{i} + 5\vec{j}$ $\vec{v} = 3\vec{i} + 7\vec{j}$

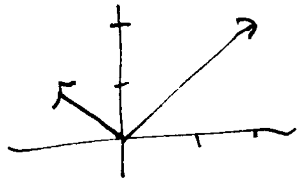
$$\begin{aligned}\vec{u}\cdot\vec{v} &= (2\vec{i}+5\vec{j})\cdot(3\vec{i}+7\vec{j}) \\ &= (2)(3)\,\vec{i}\cdot\vec{i} + (2)(7)\,\cancel{\vec{i}\cdot\vec{j}}^{\,0} \\ &\quad + (5)(3)\,\cancel{\vec{j}\cdot\vec{i}}^{\,0} + (5)(7)\,\vec{j}\cdot\vec{j} \\ &= (2)(3) + (5)(7) = 6 + 35 = 41\end{aligned}$$

<u>Fact</u> [Geometric interpretation]

$$\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta$$, where θ = angle <u>between</u> $\vec{u}$ and $\vec{v}$

<u>the idea of proof</u>: Based on the Law of Cosines.

ex :



$\vec{v} = \langle 2, 2 \rangle$

$\vec{w} = \langle -1, 1 \rangle$

$\vec{v} \cdot \vec{w} = (2)(-1) + (2)(1) = 0$

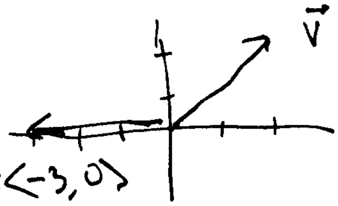
Also

$|\vec{v}|\,|\vec{w}| \cos\theta = (2\sqrt{2})(\sqrt{2}) \cancel{\cos 90^\circ}^{\,0} = 0$

Remark : If $|\vec{u}| \neq 0$ and $|\vec{v}| \neq 0$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$$

ex : Find the angle between $\vec{v} = \langle 2, 2 \rangle$ and $\vec{w} = \langle -3, 0 \rangle$



$\vec{v} = \langle 2, 2 \rangle$

$\vec{w} = \langle -3, 0 \rangle$

$|\vec{v}| = \sqrt{2^2 + 2^2} = 2\sqrt{2}$

$|\vec{w}| = \sqrt{(-3)^2 + 0^2} = 3$

$\vec{v} \cdot \vec{w} = (2)(-3) + (2)(0)$
$= -6$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|\,|\vec{w}|} = \frac{-6}{(2\sqrt{2})(3)} = \frac{-6}{6\sqrt{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

So $\theta = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = 135^\circ$

Remark : (1) If $\vec{v} \cdot \vec{w} > 0$, then $\theta < 90^\circ$

(2) If $\vec{v} \cdot \vec{w} = 0$, then $\theta = 90^\circ$

(3) If $\vec{v} \cdot \vec{w} < 0$, then $\theta > 90^\circ$.

Read the rest of the section to understand projections of vectors.

Brief review of 10.2

Solve

10.2 8)
$$\begin{cases} x + y - 3z = 8 \\ y - 3z = 5 \\ \boxed{z = -1} \end{cases}$$

we're ready for back-substitution.

$$y - 3(-1) = 5$$
$$y + 3 = 5 \Rightarrow \boxed{y = 2}$$
$$x + 2 - 3(-1) = 8$$
$$x + 5 = 8 \Rightarrow \boxed{x = 3}$$
$$\boxed{(x, y, z) = (3, 2, -1)}$$

19) Do Gaussian elimination on the system

$$\begin{cases} (1) & x + y + z = 4 \\ (2) & x + 3y + 3z = 10 \\ (3) & 2x + y - z = 3 \end{cases}$$

$-1 \cdot (1)$: $-x - y - z = -4$

(2) : $x + 3y + 3z = 10$

$$2y + 2z = 6$$

$(2')$ so $y + z = 3$

Now take

$$-2\cdot (1): \quad -2x - 2y - 2z = -8$$
$$(3): \quad 2x + y - z = 3$$
$$-y - 3z = -5$$

(3′) So $y + 3z = 5$

New, equivalent system:

$$\begin{array}{ll} (1) & x + y + z = 4 \\ (2′) & y + z = 3 \\ (3′) & y + 3z = 5 \end{array}$$

$$-1\cdot (2′): \quad -y - z = -3$$
$$(3′): \quad y + 3z = 5$$
$$2z = 2$$

(3″) So $z = 1$

This system is equivalent to our original system of equations:

$$\begin{array}{ll} (1) & x + y + z = 4 \\ (2′) & y + z = 3 \\ (3″) & z = 1 \end{array}$$

and we would be ready for back-substitution.

10.3 Gaussian Elimination using Matrices

Remark : The original system in §10.2 #19 can be represented by the ~~an~~ "augmented matrix"

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \\ 2 & 1 & -1 & 3 \end{bmatrix}$$

← a 3×4 matrix
3 rows
4 columns

We will do "elementary row operations" to modify the matrix into "row equivalent matrix"

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

in "row-echelon form."

This matrix then represents the system, equivalent to the original system:

$$\begin{cases} x + y + z = 4 \\ \quad\;\; y + z = 3 \\ \qquad\;\; z = 1 \end{cases}$$

and then back-substitute.

<u>Elementary Row Operations</u> come in three flavors:

(1) Add a multiple of (a copy of) one row to another.

e.g. $-2R_1 + R_3 \to R_3$ ..

(2) Multiply a row by a (nonzero) number.

e.g. $\frac{1}{2} R_2$.

(3) Swap two rows. e.g. $R_1 \leftrightarrow R_3$.

ex: 10.3 #21)

$$\begin{cases} x + y + z = 2 \\ 2x - 3y + 2z = 4 \\ 4x + y - 3z = 1 \end{cases}$$

So the corresponding augmented matrix is:

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & -3 & 2 & 4 \\ 4 & 1 & -3 & 1 \end{bmatrix}$$

TI84: $*row+(-2, Ans, 1, 2)$

$$-2R_1+R_2 \to R_2 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & 0 & 0 \\ 4 & 1 & -3 & 1 \end{bmatrix}$$

$*row+(-4, Ans, 1, 3)$

$$-4R_1+R_3 \to R_3 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -5 & 0 & 0 \\ 0 & -3 & -7 & -7 \end{bmatrix}$$

$*row(-1/5, Ans, 2)$

$$-\frac{1}{5}R_2 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & -7 & -7 \end{bmatrix}$$

$$3R_2+R_3 \to R_3 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

$*row+(3, Ans, 2, 3)$

$$-\frac{1}{7}R_3 \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$*row(-1/7, Ans, 3)$

What the heck. Let's try for Reduced-row-echelon form.

$$-1R_3+R_1 \to R_1 \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$-1 \cdot R_2+R_1 \to R_1 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We have just done Gauss-Jordan elimination

The underlying system of equations:

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 0 \\ 0x + 0y + 1z &= 1 \end{aligned}$$

OR

$$\begin{cases} x = 1 \\ y = 0 \\ z = 1 \end{cases}$$

25)

$$\begin{bmatrix} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

TI84: rref([A])

$$\begin{aligned} x &= -1 \\ y &= 5 \\ z &= 0 \end{aligned}$$

<u>Remark</u>: We have yet to see what happens in Gaussian elimination (or Gauss-Jordan elimination) when the system is <u>dependent</u> or is <u>inconsistent</u>.