10.3 Gaussian elimination (cont'd)

31)
$$\begin{cases} 2x - 3y - 9z = -5 \\ x + 3z = 2 \\ -3x + y - 4z = -3 \end{cases}$$

$$\begin{bmatrix} 2 & -3 & -9 & -5 \\ 1 & 0 & 3 & 2 \\ -3 & 1 & -4 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & -3 & -9 & -5 \\ -3 & 1 & -4 & -3 \end{bmatrix}$$

$$\begin{array}{c} -2R_1+R_2 \to R_2 \\ 3R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & -3 & -15 & -9 \\ 0 & 1 & 5 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & -3 & -15 & -9 \end{bmatrix}$$

$$3R_2+R_3 \to R_3 \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

our original system is equivalent to: (dependent system)

$$\begin{cases} x + 3z = 2 \\ y + 5z = 3 \\ 0 = 0 \end{cases}$$

Infinitely many solutions: we're free to let z = whatever.

Let $\boxed{z = t}$. But $y + 5(t) = 3 \Rightarrow \boxed{y = -5t + 3}$

and $x + 3t = 2 \Rightarrow \boxed{x = -3t + 2}$

Solution set $= \{ (x, y, z) \mid x = -3t+2,\ y = -5t+3,\ z = t;\ t \in \mathbb{R} \}$

Remark: We can get some solutions by taking $t = 0$ or 1 or 2 ...

$(x, y, z) = (2, 3, 0)$ or $(-1, -2, 1)$ or $(-4, 7, 2)$

will all be solutions.

Remark: If you ever get $[0 \;\; 0 \;\; 0 \;\; 1]$ as a row in an augmented matrix, that says $0x + 0y + 0z = 1$ or $0 = 1$.
This is false for any choice of (x, y, z) so there's no solution i.e. the system is inconsistent.

10.4 Matrix algebra

Addition: is defined for same-sized matrices

$$\underset{2\times 3}{\begin{bmatrix} 2 & 3 & 7 \\ 0 & -1 & -3 \end{bmatrix}} + \underset{2\times 3}{\begin{bmatrix} -4 & 0 & 10 \\ -2 & 4 & 11 \end{bmatrix}} = \underset{2\times 3}{\begin{bmatrix} -2 & 3 & 17 \\ -2 & 3 & 8 \end{bmatrix}}$$

$$\underset{3\times 1}{\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}} + \begin{bmatrix} \sqrt{2} \\ \pi \\ 3 \end{bmatrix} = \begin{bmatrix} 5+\sqrt{2} \\ \pi+2 \\ 0 \end{bmatrix}$$

Scalar multiplication

$$10\begin{bmatrix} 3 & -7 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 30 & -70 \\ 50 & 20 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 8 & 0 \\ 7 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 6 & -4 \\ 2 & -5 \end{bmatrix}$$

Remark: (1) A matrix of all zeros acts like the additive identity.
(2) Scalar multiplying by 1 acts like the multiplicative identity.

Properties: See textbook, there are no surprises.

Matrix multiplication

ex:

$$\underset{1\times 2}{\begin{bmatrix} 3 & 5 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} (3)(2)+(5)(7) \end{bmatrix}} = \begin{bmatrix} 6+35 \end{bmatrix} = \begin{bmatrix} 41 \end{bmatrix}$$

2×2 ← must match → 2×2

ex:

$$AB = \begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix} = \underset{2\times 2}{\begin{bmatrix} 41 & 11 \\ 58 & 29 \end{bmatrix}}$$

ex:

$$BA = \begin{bmatrix} 2 & -3 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 3 & -14 \\ 25 & 67 \end{bmatrix}$$

Matrix multiplication is NOT (in general) commutative. It is associative:

$(AB)C = A(BC)$

Just write ABC.

So $AB \neq BA$

ex:

$$\underset{1\times 3}{\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} 44 \end{bmatrix}}$$

ex:

$$\underset{3\times 1}{\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}} \underset{1\times 3}{\begin{bmatrix} 1 & 3 & 5 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 2 & 6 & 10 \\ 4 & 12 & 20 \\ 6 & 18 & 30 \end{bmatrix}}$$

Properties: [See textbook]

example:

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} HW & q & e1 & e2 & e3 & \text{bad test} & \text{final} \end{array} \\
\begin{array}{c} \text{Student 1} \\ \text{Student 2} \\ \text{Student 3} \end{array} & \begin{bmatrix} 100 & 100 & 100 & 100 & 100 & 100 & 100 \\ 50 & 50 & 50 & 50 & 50 & 50 & 50 \\ 100 & 100 & 100 & 100 & 100 & 50 & 100 \end{bmatrix}
\end{array}
\begin{bmatrix} .10 \\ .10 \\ .18 \\ .18 \\ .18 \\ .06 \\ .20 \end{bmatrix} = \begin{bmatrix} 100 \\ 50 \\ 97 \end{bmatrix}$$

10.5 Matrix Equations and Inverses of matrices

Methods:
1. Graphing
2. Substitution
3. Elimination
4. Gauss-Jordan elimination
5. Matrix equations

ex: [Example for motivation]

$$\begin{cases} 3x + 5y = 7 \\ 2x + 4y = 1 \end{cases}$$

Consider this matrix equation:

$$\underset{A}{\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}} \cdot \underset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{B}{\begin{bmatrix} 7 \\ 1 \end{bmatrix}}$$

has the form

$$AX = B$$

Idea: The 2x2 identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

If we can find a matrix A^{-1} such that $A^{-1}A = I$, behold:

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Fact: For 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

A will have an inverse A^{-1} if and only if

$ad - bc \neq 0$. [Remark: $ad-bc = \det(A)$]

If $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In our example: $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$ so $ad-bc = (3)(4) - (2)(5)$
$= 12 - 10 = 2 \neq 0$

and $A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{2} & -\frac{5}{2} \\ -\frac{2}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 2 & -5/2 \\ -1 & 3/2 \end{bmatrix}$

check: $A^{-1}A = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$= I$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \frac{1}{2}\begin{bmatrix} 4 & -5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 23 \\ -11 \end{bmatrix} = \begin{bmatrix} 23/2 \\ -11/2 \end{bmatrix}$$

So $(x, y) = \left(\frac{23}{2}, \frac{-11}{2}\right)$ and one can check this solution.

$\swarrow$ square matrix

<u>Defn</u>: An $n \times n$ matrix with 1's on the diagonal an zeros elsewhere is the <u>identity</u> matrix I_n.

<u>ex</u>.

$$\underset{I_2}{\overset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}} \underset{A}{\overset{2\times 3}{\begin{bmatrix} 5 & 2 & -7 \\ -4 & 11 & 17 \end{bmatrix}}} = \overset{2\times 3}{\begin{bmatrix} 5 & 2 & -7 \\ -4 & 11 & 17 \end{bmatrix}}$$

$$= A$$

$$\underset{A}{\overset{2\times 3}{\begin{bmatrix} 5 & 2 & -7 \\ -4 & 11 & 17 \end{bmatrix}}} \underset{I_3}{\overset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}} = \overset{2\times 3}{\begin{bmatrix} 5 & 2 & -7 \\ -4 & 11 & 17 \end{bmatrix}}$$

$$= A$$

<u>For an example</u> of how to find A^{-1} if A is an invertible 3×3 matrix, see EXAMPLE 4, Section 10.5 (p675).