3.5 Double Angle Identities (cont'd)

We know $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

is true for any two angles α and β. In particular, if both are equal to x:

$$\cos(x+x) = \cos x\cos x - \sin x \sin x$$

$$\boxed{\cos 2x = \cos^2 x - \sin^2 x}$$ (1st) Double Angle formula for Cosine

$$\cos 2x = \cos^2 x - (1-\cos^2 x)$$
$$= \cos^2 x - 1 + \cos^2 x$$
$$\boxed{\cos 2x = 2\cos^2 x - 1}$$ (2nd) Double Angle formula

HUGELY important in calculus!!

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= (1-\sin^2 x) - \sin^2 x$$
$$\boxed{\cos 2x = 1 - 2\sin^2 x}$$ (3rd) Double Angle formula for Cosine

ex: Let $x = \frac{\pi}{6}$ with 3rd formula

$$\cos\left(2\tfrac{\pi}{6}\right) = \cos\frac{\pi}{3} = \cos 60^\circ = \frac{1}{2} = LHS$$

$$1 - 2\sin^2\frac{\pi}{6} = 1 - 2\left(\sin\frac{\pi}{6}\right)^2$$
$$= 1 - 2\cdot\left(\frac{1}{2}\right)^2 = 1 - 2\cdot\left(\frac{1}{4}\right)$$
$$= 1 - \frac{1}{2} = \frac{1}{2} = RHS$$

ex: Prove the identity

$$\cos(3x) = \cos^3 x - 3\cos x \sin^2 x$$

sum ID for cos

$$LHS = \cos(2x + x) \overset{\text{sum ID for cos}}{=} \cos 2x \cos x - \sin 2x \sin x$$

double angle ID for cos and sin

$$= (\cos^2 x - \sin^2 x)\cos x - (2\sin x \cos x)\sin x$$

$$= \cos^3 x - \cos x \sin^2 x - 2\cos x \sin^2 x$$

$$= \cos^3 x - 3\cos x \sin^2 x = RHS \quad \text{and we're done.}$$

ex Derive the double angle ID for tangent:

double angle IDs

$$\tan 2x \overset{\text{quotient ID}}{=} \frac{\sin 2x}{\cos 2x} \overset{\text{double angle IDs}}{=} \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x} \cdot \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x}}$$

quotient ID

$$= \frac{\frac{2\sin x \cos x}{\cos^2 x}}{\frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{2\sin x}{\cos x}}{1 - \left(\frac{\sin x}{\cos x}\right)^2} \overset{\text{quotient ID}}{=} \frac{2\tan x}{1 - \tan^2 x}$$

Half-angle identities

Take $\cos 2x = 1 - 2\sin^2 x$ and solve for $\sin x$:

$$-1 + \cos 2x = -1 + 1 - 2\sin^2 x$$

$$-1 + \cos 2x = -2\sin^2 x \qquad \text{Multiply by } -\tfrac{1}{2}:$$

$$-\frac{1}{2}(-1 + \cos 2x) = \left(-\frac{1}{2}\right)\cdot(-2)\sin^2 x$$

$$\frac{1 - \cos 2x}{2} = \sin^2 x$$

$$\sin x = \pm\sqrt{\frac{1 - \cos 2x}{2}}$$

Let $2x = u$ so that $x = \frac{u}{2}$.

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}}$$

But u is just a letter. So is x. Use x.

Half angle IDs for sine and cosine

$$\boxed{\sin\frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}}$$

similarly:

$$\boxed{\cos\frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{2}}}$$

ex. What is the exact value of $\sin 22.5°$?

Hint: $22.5° = \frac{45°}{2}$

$$\sin 22.5° = \sin\frac{45°}{2} = \pm\sqrt{\frac{1-\cos 45°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}\cdot\frac{2}{2}}$$

$$= \pm\sqrt{\frac{2-\frac{2\sqrt{2}}{2}}{4}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{\sqrt{4}}$$

$$= \pm\frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

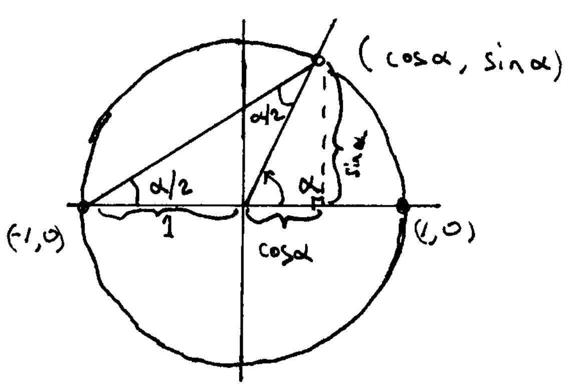
which? '+' because sine is positive in Q I

Three more HALF angle identities, for tangent:

$$\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

Added after class:

Graphical interpretation of $\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$.



Algebraic derivation of this identity:

$$\frac{\sin\alpha}{1+\cos\alpha} = \frac{\sin 2(\frac{\alpha}{2})}{1+\cos 2(\frac{\alpha}{2})} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{1+2\cos^2\frac{\alpha}{2}-1}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

In the figure,

$$\tan\frac{\alpha}{2} = \frac{\text{opp}}{\text{adj}} = \frac{\sin\alpha}{1+\cos\alpha}$$