6.1 Complex numbers (review)

$i = \sqrt{-1}$ so $i^2 = -1$

ex. (of complex number) has form $a+bi$

$3+4i$; $2+\frac{1}{2}i$, $-\frac{7}{4}+\frac{\sqrt{3}}{2}i$;

$17 = 17+0i$, $-3i = 0-3i$.

ex. [Add] $(3+4i)+(7+2i) = 10+8i$

ex. (Multiply) $(3+4i)(2+i) = 6+3i+8i+4i^2$

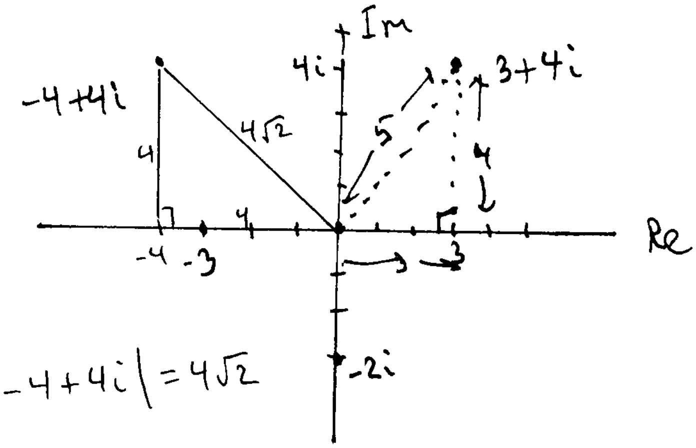
Now use $i^2=-1$: $= 6+11i+4(-1)$

$= 2+11i$

ex: (Divide): $\frac{8-i}{2+i} \cdot \frac{2-i}{2-i} = \frac{16-8i-2i+i^2}{2^2-i^2} = \frac{16-10i+(-1)}{4-(-1)}$

$= \frac{15-10i}{5} = \frac{15}{5} - \frac{10i}{5} = 3-2i$

6.2 Complex plane and Polar form of a complex number



$|-4+4i| = 4\sqrt{2}$

Defn. The absolute value or modulus of a complex number $z = a+bi$ is

$$|z| = \sqrt{a^2+b^2}$$

ex: $|3+4i| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

ex: $|-3| = |-3+0i| = \sqrt{(-3)^2+0^2}$

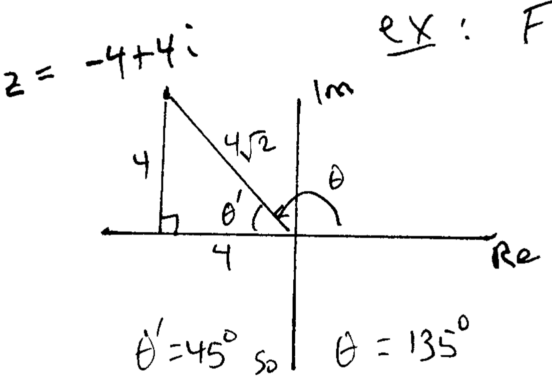
$= \sqrt{9} = 3$

Defn: The trig form (or polar form) of a complex number

is $\boxed{z = r(\cos\theta + i\sin\theta)}$

where $r = |z| = \sqrt{a^2+b^2}$ and

θ = angle in standard position with terminal ray through $a+bi$.



ex: Find the trig form of $z = -4+4i$.

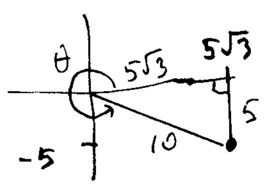
$r = |z| = \sqrt{(-4)^2+4^2} = \sqrt{32} = 4\sqrt{2}$

$\theta = 135^\circ$

$z = -4+4i = \boxed{4\sqrt{2}\left(\cos 135^\circ + i\sin 135^\circ\right)}$

$= 4\sqrt{2}\ \text{cis}\ 135^\circ$

ex: Find trig of $5\sqrt{3} - 5i = a+bi$ where $a = 5\sqrt{3}$, $b = -5$



Use $\boxed{\begin{array}{l} r = \sqrt{a^2+b^2} \\ \tan\theta = \frac{b}{a} \end{array}}$ ← but choose the correct quadrant

$r = \sqrt{(5\sqrt{3})^2 + (-5)^2} = \sqrt{75+25} = \sqrt{100} = 10$

$\tan\theta = \frac{b}{a} = \frac{-5}{5\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

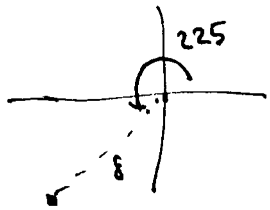
So $\theta = \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -30^\circ$ OR $-30^\circ + 360^\circ = 330^\circ$

$z = 5\sqrt{3} - 5i = 10(\cos 330^\circ + i\sin 330^\circ)$

standard form ↑   ↑ polar or trig form

<u>ex</u>: [Trig form → std form of a complex number]

$$z = 8(\cos 225^\circ + i\sin 225^\circ)$$



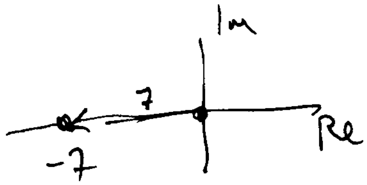
$$= 8\left[(-\cos 45^\circ) + i\,(-\sin 45^\circ)\right]$$

$$= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] = -\frac{8\sqrt{2}}{2} - \frac{8\sqrt{2}}{2}i$$

$$= -4\sqrt{2} - 4\sqrt{2}\,i$$

<u>ex</u>: $z = 7(\cos 180^\circ + i\sin 180^\circ)$

$$= 7(-1 + 0i) = -7 + 0i = -7$$



<u>Fact</u>: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then

(1) $z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$

[In words: <u>Multiply</u> the moduli and <u>add</u> the arguments.]

Also (2) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

[In words: <u>Divide</u> the moduli and <u>subtract</u> the arguments.]

ex: $z_1 = 12(\cos 75^\circ + i\sin 75^\circ)$

$z_2 = 3(\cos 30^\circ + i\sin 30^\circ)$

$z_1 z_2 = 36(\cos 105^\circ + i\sin 105^\circ)$

$\dfrac{z_1}{z_2} = 4(\cos 45^\circ + i\sin 45^\circ)$

6.2 49) Let $z_1 = 2 \text{ cis } 150°$ and $z_2 = 3 \text{ cis } 300°$

Find $z_1 \cdot z_2$ and write in <u>standard</u> form.

$$z_1 z_2 = 2 \cdot 3 \left[\text{cis} \left(150° + 300°\right)\right]$$
$$= 6 \text{ cis } 450°$$
$$= 6 (\cos 450° + i \sin 450°)$$
$$= 6 (\cos 90° + i \sin 90°)$$
$$= 6 (0 + i \cdot 1) = 6i$$

Note: $450° = 360° + 90°$

6.3 Powers and Roots of complex numbers

<u>Remark</u>: The angle θ is <u>not</u> unique in the trig form of a complex number. We can add or subtract $360° k$.

<u>ex</u>: $z = 6 (\cos 90° + i \sin 90°) = 6i$

Find z^3, using the <u>trig</u> form.

$$z^3 = z \cdot z \cdot z = 6 \cdot 6 \cdot 6 \left[\cos (90° + 90° + 90°) + i \sin (90° + 90° + 90°)\right]$$
$$= 6^3 \left[\cos (3 \cdot 90°) + i \sin (3 \cdot 90°)\right]$$
$$= 216 (\cos 270° + i \sin 270°) \leftarrow \text{answer in trig form}$$
$$= 216 (0 + i (-1))$$
$$= -216 i \leftarrow \text{answer in std form,}$$

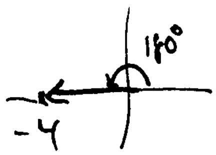
<u>ex</u>. $z = \sqrt{2} (\cos 45° + i \sin 45°)$ Find z^4.

$$z^4 = \sqrt{2}^4 \left[\cos (4 \cdot 45°) + i \sin (4 \cdot 45°)\right]$$
$$= \boxed{4 (\cos 180° + i \sin 180°)} = \boxed{-4}$$

That is, $\sqrt{2} (\cos 45° + i \sin 45°) = 1 + i$ is a 4^{th} root of -4.

ex: Find all four 4^{th} roots of -4.

$$-4 = -4 + 0i = 4(\cos 180^\circ + i \sin 180^\circ)$$



More generally,

$$z = -4 = 4\left[\cos(180^\circ + k360^\circ) + i \sin(180^\circ + k360^\circ)\right]$$

To find the 4^{th} roots of z

① Find the 4^{th} root of the modulus 4

$$\sqrt[4]{4} = 4^{1/4} = (2^2)^{1/4} = 2^{2/4} = 2^{1/2} = \sqrt{2}$$

② Divide the argument by 4:

$$\frac{180^\circ + k360^\circ}{4} = \frac{180^\circ}{4} + k\frac{360^\circ}{4} = 45^\circ + k90^\circ$$

4^{th} roots:

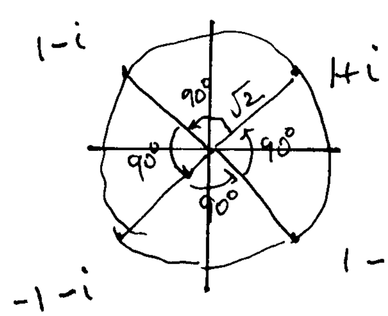
Take $k=0$: $w_0 = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ) = 1+i$

Take $k=1$: $w_1 = \sqrt{2}(\cos 135^\circ + i \sin 135^\circ) = -1+i$

$k=2$: $w_2 = \sqrt{2}(\cos 225^\circ + i \sin 225^\circ) = -1-i$

$k=3$: $w_3 = \sqrt{2}(\cos 315^\circ + i \sin 315^\circ) = 1-i$

$[w_4 = w_0$
$w_5 = w_1]$



Final part 1 - is due Wed. and covers Ch 5 and 6. It is 1/4 of your final score.

Final part 2 - is in class Wed. and covers Ch 1-4. It is 3/4 of your final score. No notes.

You have at least another week for homework.