1. Draw the isoclines for the differential equation for \( c \) in \{-2, -1, 0, 1, 2\}. Sketch the Direction Field and some solution curves.

\[
\frac{dy}{dx} = \frac{y}{x}
\]

- \( c = 0 \): \( \frac{y}{x} = 0 \) » \( y = 0 \) (\( x \neq 0 \))
- \( c = 1 \): \( \frac{y}{x} = 1 \) » \( y = x \)
- \( c = 2 \): \( \frac{y}{x} = 2 \) » \( y = 2x \)
- \( c = -1 \): \( \frac{y}{x} = -1 \) » \( y = -x \)
- \( c = -2 \): \( \frac{y}{x} = -2 \) » \( y = -2x \)

The direction field segments are the same as the isoclines except that \( x \neq 0 \).

2. Is a unique solution guaranteed for the following IVP? Explain your answer.

\[
\frac{dy}{dx} = \frac{y}{x+1}, \quad y(0) = 2
\]

Here, \( f(x,y) = \frac{y}{x+1} \) is continuous as long as \( x \) is not close to -1. So, we can consider a solution within the rectangle with \(-1/2 < x < 1/2\) and \( 0 < y < 4 \).

\( D_y(f) = \frac{1}{x+1} \) which is also continuous within the rectangle specified. Therefore, we are guaranteed a unique solution to this IVP.

Classify & solve each differential equation:

3. \((x^2 + 2y^2)dx + xydy = 0 \) » » \[
\frac{dy}{dx} = - \frac{x^2 + y^2}{xy} = - \frac{x}{y} - 2\frac{y}{x}
\]

Homogeneous first order: let \( v = \frac{y}{x} \) or \( y = vx \) & \( \frac{dy}{dx} = x\frac{dv}{dx} + v \)

» » \[
\frac{dv}{dx} + v = - \frac{1}{v} - 2v \quad \Rightarrow \quad \frac{dv}{dx} = - \frac{3v^2 + 1}{v} \quad \Rightarrow \quad \frac{v}{3v^2 + 1} dv = - \frac{1}{x} dx
\]

» » Making the substitution \( u = 3v^2 + 1, du = 6vdv \) » » \[
\frac{1}{6} \ln(3v^2 + 1) = - \ln|x| + C
\]
\[ \ln(3v^2 + 1) = -\ln|x| + \hat{C} \]  
\[ \ln(3v^2 + 1) = \ln|\hat{C} x^6| \]  
\[ 3v^2 + 1 = \hat{C} x^6 \]  
\[ 3(\frac{y}{x})^2 + 1 = \hat{C} x^6 \]  
\[ 3y^2 + x^2 = \hat{C} x^4 \]

4. \( \frac{dy}{dx} = (2x + 4y + 8)^2 \) Random Substitution. Let \( v = 2x + 4y + 8 \) then
\[
\frac{dv}{dx} = 2 + 4\frac{dy}{dx}
\]
\[
\frac{1}{4}\frac{dv}{dx} - \frac{1}{2} = v^2 \quad \frac{dv}{dx} = 4v^2 + 2 \quad \frac{dv}{4v^2+2} = dx
\]
\[
\int \frac{dv}{4v^2+2} = \int dx \quad \text{Subst. } u = 2v, du = 2dv \quad \frac{1}{2} \int \frac{du}{u^2+2} = \int dx
\]
\[
\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) = x + C \quad \frac{1}{2}\sqrt{2}\tan^{-1}\left(\frac{2v}{\sqrt{2}}\right) = x + C
\]
\[
\tan^{-1}\left(\frac{2v}{\sqrt{2}}\right) = 2\sqrt{2}x + \hat{C} \quad \frac{2v}{\sqrt{2}} = \tan(2\sqrt{2}x + \hat{C}) \quad \sqrt{2}v = \tan(2\sqrt{2}x + \hat{C})
\]
\[
2x + 4y + 8 = \frac{1}{\sqrt{2}}\tan(2\sqrt{2}x + \hat{C}) \quad 4y = \frac{1}{\sqrt{2}}\tan(2\sqrt{2}x + \hat{C}) - 2x - 8
\]
\[
y = \frac{1}{4\sqrt{2}}\tan(2\sqrt{2}x + \hat{C}) - \frac{1}{2}x - 2
\]

5. \( \frac{dy}{dx} + 2xy = x^3 \) Linear first order
\[
\text{IF} = e^{\int 2xdx} = e^{x^2} \quad \text{Multiplying the equation}
\]
\[
e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y = x^3e^{x^2} \quad d(e^{x^2}y) = x^3e^{x^2}
\]
\[
\int d(e^{x^2}y) = \int x^3e^{x^2}dx = \int x^2xe^{x^2}dx \quad (*)
\]

Integrating the right side by parts with \( dv = xe^{x^2}dx \) & \( u = x^2 \)

which yields \( v = \frac{1}{2}e^{x^2} \) & \( du = 2xdx \)

\( (*) \) becomes \( ye^{x^2} = \frac{x^2}{2}e^{x^2} - \int xe^{x^2}dx = \frac{x^2}{2}e^{x^2} - \frac{1}{2}e^{x^2} + C \)
\[
y = \frac{x^2}{2} - \frac{1}{2} + Ce^{-x^2} = \frac{1}{2}(x^2 - 1) + Ce^{-x^2}\]
6. \( xdy = (y + 2xe^{y/x}) \)

Rewrite: \( \frac{dy}{dx} = \frac{y}{x} + 2e^{y/x} \) Homogeneous First Order

Let \( v = \frac{y}{x}, y = vx, \frac{dv}{dx} = x\frac{dy}{dx} + v \)

\( \Rightarrow x\frac{dv}{dx} + v = v + 2e^{-v} \Rightarrow x\frac{dv}{dx} = 2e^{-v} \Rightarrow e^{v}dv = \frac{2}{x}dx \Rightarrow e^{v} = 2lnx + C \)

\( \Rightarrow e^{v} = ln(x^{2} + C) \Rightarrow e^{y/x} = ln(x^{2} + C) \) Applying ln to both sides

\( \Rightarrow \frac{v}{x} = ln(ln(x^{2} + C)) \Rightarrow y = xln(ln(x^{2} + C)) \)

7. \( \frac{dy}{dx} + (cscx)y = sinx \)

Linear First Order

\( IF = e^{\int cscxdx} = e^{\ln |cscx - cotx|} = cscx - cotx \)

Multiplying through by the IF:

\( (cscx - cotx)\frac{dy}{dx} + (csc^{2}x - cscxcotx)y = 1 - cosx \)

\( \Rightarrow d((cscx-cotx)y) = 1 - cosx \)

\( \Rightarrow \int d((cscx - cotx)y) = \int (1 - cosx)dx \)

\( \Rightarrow (cscx - cotx)y = x - sinx + C \Rightarrow y = \frac{x - sinx + C}{cscx - cotx} \) or \( y = \frac{sin(x - sinx + C)}{1 - cosx} \)

8. We begin with the differential equation: \( dV/dt = 2 \) liters/min (the soln rate in - rate out)

\( dV = 2dt \Rightarrow V = 2t + C \) Applying the IC: \( V(0) = 100 \) liters we see that \( C = 100 \).

\( V(t) = 2t + 100 \)

Now, we can write the differential equation \( dX/dt = \) rate of solute in - rate of solute out (in kg/min)

\[ \frac{dX}{dt} = \frac{16 \text{ kg}}{\text{min}} - \frac{6 \text{ liters}}{2t+100 \text{ liters}} \times \frac{X \text{ kg}}{2t+100 \text{ liters}} = \frac{16 \text{ kg}}{\text{min}} - \frac{3X \text{ kg}}{t+50 \text{ min}} = 16 - \frac{3X}{t+50} \]

\[ \frac{dX}{dt} + \frac{3}{t+50}X = 16 \text{ Which is First Order Linear} \]

\( IF = e^{\int \frac{3}{t+50}dt} = e^{3\ln|t+50|} = (t + 50)^{3} \Rightarrow \) The de becomes \( d(X(t+50)^{3}) = 16(t+50)^{3} \) which is readily integrated \( \Rightarrow X(t+50)^{3} = 4(t+50)^{4} + C \) or \( X = 4(t+50) + C(t+50)^{-3} \)

Applying the IC \( X(0) = 20 \) kg, we find that \( C = -22,500,000 \)

so that \( X(t) = 4(t+50) - 2.25x10^{7}(t+50)^{-3} \)

And, \( X(30 \text{ mins}) = 4(30+50) - 2.25x10^{7}(30+50)^{-3} \approx 276.06 \) kg