Solving Systems by Gaussian Elimination

Gaussian Elimination is a streamlined method of Elimination that was discovered by Gauss.

Let us first look at the desired form

For a consistent second order system, the goal is:
\[
\begin{bmatrix}
1 & \# & \#
\end{bmatrix}
\] **ones are ideal and can be obtained at the last step**

For a consistent third order system, the goal is:
\[
\begin{bmatrix}
1 & \# & \# \\
0 & 1 & \#
\end{bmatrix}
\] **zeros are necessary for the method to work**

The valid operations that can be performed are called Elementary Row Operations:

1. You can multiply any row by a nonzero constant (Can multiply or divide)
2. You can interchange any two rows. (Nice to use to get a 1 in the right row)
3. You can add a multiple of one row to another row. (Most often used to get the 0’s)

Performing any of the 3 operations above is valid. In using these operations you will want to have a strategy.

1. It is best to work one column at a time, working from left to right. For instance, in a third order system you want to get column one to look like:
\[
\begin{bmatrix}
1 & \# & \\
0 & 1 & \\
0 & 0 & 1
\end{bmatrix}
\]

2. Within each column you want to use the diagonal entry to make the zeros below. If the diagonal entry is one it is easy to use Elementary Row Op #3 to make the zeros.

Example:

\[
\begin{align*}
2x + 3y - z &= 4 \\
x - 2y + 3z &= 2 \\
3x + y + 4z &= 8
\end{align*}
\]

Form the matrix:
\[
\begin{bmatrix}
2 & 3 & -1 & 4 \\
1 & -2 & 3 & 2 \\
3 & 1 & 4 & 8
\end{bmatrix}
\]

Interchange with \( R_1 \) Interchange with \( R_2 \):
\[
\begin{bmatrix}
1 & -2 & 3 & 2 \\
2 & 3 & -1 & 4 \\
3 & 1 & 4 & 8
\end{bmatrix}
\]

\(-2R_1\): 
\[
\begin{bmatrix}
-2 &  4 & -6 & -4
\end{bmatrix}
\]

\(-3R_1 + R_2\) becomes the new row 2 with the \(-2R_1\) being done mentally or as scratch as shown above. The new matrix becomes:
\[
\begin{bmatrix}
1 & -2 & 3 & 2 \\
0 & 8 & -7 & 0 \\
3 & 1 & 4 & 8
\end{bmatrix}
\]

We still need to get a zero where the 3 is in column 1. We use the diagonal entry again and Elementary Row Op #3. The Operation will be \(-3R_1 + R_3\) which will become the new Row 3.

As before we will do the \(-3R_1\) as scratch work above the matrix:
\[
\begin{bmatrix}
-3 &  6 & -9 & -6
\end{bmatrix}
\]

\(-3R_1 + R_3\):
\[
\begin{bmatrix}
-2 & 3 & 2 \\
0 & 7 & -7 & 0 \\
3 & 1 & 4 & 8
\end{bmatrix}
\]

\(-3R_1 + R_3\): 
\[
\begin{bmatrix}
1 & -2 & 3 & 2 \\
0 & 7 & -7 & 0 \\
0 & 7 & -5 & 2
\end{bmatrix}
\]

We are done with Column #1 and move to Column #2.
We would like the diagonal entry in Col #2 to be a one. Here we can multiply Row 2 by 1/7 to make the diagonal entry a 1. This is utilizing Elementary Row Op # 1

$$(1/7)\text{ times Row 2: } \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & -5 & 2 \end{bmatrix}$$

Now we can use the diagonal entry to make a zero in the bottom row, second column. We will perform the Elementary Row Op: -7R₂ + R₃ to make the new Row 3. Again we will do the -7R₂ as scratch above the matrix:

$$-7R₂: \begin{bmatrix} 0 & -7 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & -5 & 2 \end{bmatrix} - 7 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Finally, we move to the third column. To make the diagonal entry a 1 we can multiply that row by 1/2. We do this in the matrix as we want a 1 there.

$$(1/2)R₃: \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Now the matrix is in the ideal form so we can put the variables back into the problem starting with the bottom row. In doing this we are writing the system that is equivalent to this matrix. Notice that the zeros indicate eliminated variables.

Row3: $$0x + 0y + 1z = 1$$ Or, just $$z = 1$$.
Row2: $$0x + 1y - 1z = 0$$ Or, $$y - z = 0$$. Here we can substitute $$z = 1$$ and solve for $$y$$.

$$y - 1 = 0 \quad \Rightarrow \quad y = 1$$

Row1: $$1x - 2y + 3z = 2$$ We can substitute $$y = 1$$ and $$z = 1$$ and solve for $$x$$.

$$x - 2(1) + 3(1) = 2 \quad \Rightarrow \quad x - 2 + 3 = 2 \quad \Rightarrow \quad x = 1$$

The solution is $$(1, 1, 1)$$.

Ex: Solve the system using Gaussian Elimination.

$$2x + 5y + 3z = 15$$
$$6x - 3y + z = 11$$
$$4x + 3y - 2z = 7$$

(In this example making one as the pivot is not recommended at the beginning since the 2x can be readily used to make the 6x & 4x zero using elementary row operation 3)

Ex: Solve the system using Gaussian Elimination.
\[4x + 3y + 2z = 14\]
\[7x - 2y + 2z = 10\]
\[13x + 2y - z = 22\]

(In this example it is possible to get a one as your pivot in the first column by one of the following operations. \(-5R_1 + 3R_2\) making this the new Row 1 OR \(-3R_1 + R_3\) making this the new Row 1)

Ex: Student Loans
Stacey’s two student loans totaled $12,000. One of her loans was at 6.5% simple interest and the other at 7.2% interest. After one year, Stacey owed $811.50 in interest. What was the amount of each loan?
Formula: \(Prt = I\)

Ex: Food Science or Percent Concentration
The milk used for cream cheese is ideally 8% milk fat. Your objective is to mix Whole Milk which is 4% milk fat and Cream which is 28% milk fat to make 200 lb of milk for cream cheese.
This is called a Total Value Mixture Problem
Formula:
\[\text{Amount of solution} \times \text{Percent Concentration} = \text{Quantity of Pure Substance} \quad \text{or} \quad A \times P = Q\]

Basic set up:
Total Value of the milk in the final product is to be 200 lb.
This means that you can let \(x = \# \text{ lbs of Whole Milk}\) and \(y = \# \text{ lbs of Cream}\) and \(x + y = 200\)

<table>
<thead>
<tr>
<th>Amount of Solution</th>
<th>*</th>
<th>Percent Concentration</th>
<th>=</th>
<th>Quantity of Pure Milk Fat</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>*</td>
<td>0.04</td>
<td>=</td>
<td>0.04x</td>
</tr>
<tr>
<td>(y)</td>
<td>*</td>
<td>0.28</td>
<td>=</td>
<td>0.28y</td>
</tr>
<tr>
<td>Mixture: (x + y = 200)</td>
<td></td>
<td></td>
<td></td>
<td>Pure Milk Fat in Mixture: 0.04x + 0.28y = 200(0.08)</td>
</tr>
</tbody>
</table>

System: \(x + y = 200\) (Total amount of cream cheese made in the mixture)
\[0.04x + 0.28y = 16\] (Total amount of milk fat in the mixture)

Ex: Total Value Problem
Julie, a rather strange graduate student, prefers snakes as her favorite pets. On one trip to her favorite exotic pet store she purchases 2 snakes, 4 pink rats and 5 fuzzy rats for a total of $43.65 excluding tax. The next week she realizes she needs more snake food so she makes another trip to the pet store. On this trip she cannot resist and purchases one last snake along with the intended snake food. With the new snake she purchases 12 pink rats and 15 fuzzy rates for $38.45. A few weeks later Julie finds that she is in need of more snake food. This time she decides to stock up and purchases 20 pink rats and 30 fuzzy rats for $37.50. Find the cost that Julie pays for each snake, pink rat and fuzzy rat at her favorite pet store.
Here you can let \(s = \text{the cost of a snake, } p = \text{the cost of a pink rat and } f = \text{the cost of a fuzzy rat.}\)

Each trip to the pet store represents one equation:
Trip 1: \(2s + 4p + 5f = 43.65\)
Trip 2: \(s + 12p + 15f = 38.45\)
Trip 3: \(20p + 30f = 37.50\)

What is the best order to place your rows in the matrix to save yourself some work?