Solving Exponential and Logarithmic Equations

Exponential Equations

- **Type I** – Both sides may be expressed in terms of a common base
  
  Ex: \( 2^x = \frac{1}{8} \Rightarrow 2^x = 2^{-3} \Rightarrow x = -3 \) (since this is a one-to-one function)
  
  Ex: \( 4^{x-2} = 2^{2x+1} \Rightarrow 2^{2(x-2)} = 2^{2x+1} \Rightarrow 2x - 4 = x + 1 \Rightarrow x = 5 \)

- **Type II** – Both sides are not readily expressed in terms of a common base.
  
  In these problems we apply a logarithm to both sides so that we can then use the power rule for logarithms to bring the powers out front. If the problem contains e as a base it is best to use ln. Otherwise you can use log or ln.
  
  Ex: \( 5^x = 12 \Rightarrow \log(5^x) = \log(12) \Rightarrow x \log(5) = \log(12) \)
      \[ x = \frac{\log(12)}{\log(5)} \approx 1.544 \]  
      (Since x is in the exponent we need to carry more decimal places than usual as the rounding error can have an extreme effect)

  Ex: \( 2^{-x+1} = 12^x \Rightarrow \log(2^{-x+1}) = \log(12^x) \Rightarrow (x-1)\log(2) = x\log(12) \)
      \[ x = \frac{\log(12)}{\log(2) - \log(12)} \approx -0.387 \]  
      (Again, since x is in the exponent we need to carry more decimal places than usual as the rounding error can have an extreme effect)

  Ex: \( e^{4x} = 1,280 \) {Use ln this time}
      \[ \ln(e^{4x}) = \ln(1,280) \Rightarrow 4x \ln(e) = \ln(1,280) \Rightarrow 4x = \ln(1,280) \]
      \[ x = \frac{\ln(1,280)}{4} \approx 1.7887 \]

  Ex: \( 8,000e^{0.02t} = 24,000 \) {First divide by 8,000 to get e isolated}
      \[ e^{0.02t} = 3 \Rightarrow \ln(e^{0.02t}) = \ln(3) \Rightarrow 0.02t \ln(e) = \ln(3) \]
      \[ 0.02t = \ln(3) \Rightarrow t = \frac{\ln(3)}{0.02} \approx 54.931 \]
Logarithmic Equations
The first step is to condense both sides when possible. Sometimes terms must be
moved to the other side in order to achieve this. Then, get rid of the logarithm by
using the definition of the logarithm or the one-to-one property. Solve the resulting
equation. Finally, check that all solutions are in the domain of the original equation.
Do not forget that the argument of a logarithm must be nonnegative!

Ex: \( \log_{2}(2x-4) = 3 \) \{Since only one log, no need to condense\}
\[ \Rightarrow 2^3 = 2x - 4 \quad \{\text{the equivalent exponential form}\} \]
\[ \Rightarrow 8 = 2x - 4 \quad \Rightarrow 12 = 2x \quad \Rightarrow 6 = x! \]
Check that 6 is in the domain: the only argument is 2x - 4 \( \Rightarrow 2(6) - 4 = 12 - 4 = 8 \)
which is valid! \( \Rightarrow \) The solution is 6.

Ex: \( \log_{2}x + \log_{2}(x-6) = 4 \) \{Condense the left side\}
\[ \Rightarrow \log_{2}(x^2 - 6x) = 4 \quad \Rightarrow 2^4 = x^2 - 6x \quad \Rightarrow 16 = x^2 - 6x \quad \Rightarrow 0 = x^2 - 6x - 16 \]
\[ \Rightarrow 0 = (x-8)(x+2) \quad \Rightarrow x = 8 \text{ or } x = -2 \quad \{\text{Check domain}\} \]
\[ \Rightarrow x = 8 \text{ is the only solution.} \]

Ex: \( \log_{3}x - \log_{3}(x-6) = \log_{3}(8) \)
\[ \Rightarrow \log_{3}\left(\frac{x}{x-6}\right) = \log_{3}(8) \]
\[ \Rightarrow \frac{x}{x-6} = 8 \quad \{\text{The one-to-one property}\} \]
\[ \Rightarrow 8(x-6) = x \quad \{\text{Set the cross products equal since it is a proportion}\} \]
\[ \Rightarrow 8x - 48 = x \]
\[ \Rightarrow 7x = 48 \]
\[ \Rightarrow x = \frac{48}{7} \approx 6.86 \quad \{\text{Check that it is in the domain of the original equation – it is!}\} \]

Ex: \( 2 - \log_{4}(x-15) = \log_{4}(x) \)
\{Note that you cannot condense as is – move the logs to the same side so that you
can condense\}
\[ \Rightarrow 2 = \log_{4}(x-15) + \log_{4}(x) \]
\[ \Rightarrow 2 = \log_{4}(x^2 - 15x) \quad \{\text{Condense}\} \]
\[ \Rightarrow 4^2 = x^2 - 15x \quad \{\text{Use the definition of the logarithm}\} \]
\[ \Rightarrow 16 = x^2 - 15x \quad \Rightarrow 0 = x^2 - 15x - 16 \]
\[ \Rightarrow 0 = (x-16)(x+1) \quad \Rightarrow x = 16 \text{ or } x = -1 \]
Only 16 is in the domain of the original equation so \( \Rightarrow x = 16 \)